The Fundamental Theorem of the Phase Line

Theorem 1. *Suppose that* $-\infty < a < b < \infty$ *and* $f(x)$ *is a continuously differentiable function on a neighborhood of* $[a, b]$*, with*

$$f(a) = f(b) = 0, \qquad \text{and,} \qquad f(x) > 0 \quad \text{for} \quad a < x < b\,.$$

Then for any $a < x_0 < b$ *the solution of the initial value problem*

$$x' = f(x), \qquad x(0) = x_0$$

i. *Satisfies* $a < x(t) < b$*.*

ii. *Exists for* $-\infty < t < \infty$*.*

iii. *Is strictly monotone increasing.*

iv. *Satisfies*

$$\lim_{t\to\infty} x(t) = b, \qquad \lim_{t\to-\infty} x(t) = a\,.$$

Remark. The analogous result for $f < 0$ follows from this one on making the change of variable $s = -t$ corresponding to time reversal.

Remark. If $f(a) = 0$ and $f(x) > 0$ for all $x > a$ the conclusion is that solutions starting to the right of a are strictly increasing, approach a as $t \to -\infty$, and approaches $+\infty$ either as $t \to +\infty$ or as $t \to T_*$ where $[0, T^*)$ is the maximal interval of existence.

Proof. i. Since $x(t)$ is continuous, in order for **i.** to be violated there must be a T so that $x(T) = a$ or $x(T) = b$. The uniqueness theorem implies that if $x(T) = a$ then $x(t) = a$ for all time violating the initial condition. The possibility $x(T) = b$ is ruled out in the same way.

ii. Part **i.** shows that $x(t)$ takes values in the closed bounded set $[a, b]$. The fundamental existence theorem implies therefore, that $x(t)$ exists for all time.

iii. Since $a < x(t) < b$, one has $x' = f(x) > 0$.

iv. Suppose that $I \subset (a, b)$ is a closed interval. Suppose that $x(t) \in I$ for $t_1 \le t \le t_2$. Define

$$m \;:=\; \min_{x\in I} f(x) \;>\; 0\,.$$

Then

$$b - a > x(t_2) - x(t_1) \;=\; \int_{t_1}^{t_2} x'(s)\,ds = \int_{t_1}^{t_2} f(x(s))\,ds \ge \int_{t_1}^{t_2} m\,ds = m(t_2 - t_1)\,.$$

Thus $t_2 - t_1 < (b - a)/m$.

Taking $t_1 = 0$ shows that the solution increases beyond I before time $(b - a)/m$. This proves that $\lim_{t\to\infty} x(t) = b$. A similar argument works for the limit $t \to -\infty$. $\square$

Discussion. i. This result is usually introduced in first year calculus courses where Theorem statements and proofs are not common. The subject is usually not discussed in advanced courses. This beautiful little proof is hard to find.

ii. Note the differential inequality in bounding the time that can be spent in I. It resembles the differential inequality proof that linear systems cannot have finite time blow up. Here a lower bound on f is used while in the other it is an upper bound.