Topological Conjugacy in Dimension 1

1 Definition.

Suppose that the flow of the ordinary differential equation

\[ \dot{x} = f(x) \]  

maps the interval \( I \) to itself for \( t > 0 \). We say that \( I \) is invariant. The
interval may be open, closed, or half open. It can be infinite on one side or
both.

Similarly suppose that \( J \) is an invariant interval for the differential equation

\[ \dot{y} = g(y) \].

Denote by \( \phi(t,x) \) the flow of the \( x \) equation and \( \psi(t,y) \) the flow of the \( y \)
equation. Invariance says that for \( t \geq 0 \) and \( x \in I, \phi(t,x) \in I \). Similarly for
\( J \).

**Definition 1.1** The differential equations on \( I \) and \( J \) are topologically
conjugate when there is a continuous \( h : I \rightarrow J \) that is one to one, onto,
with continuous inverse so that for all \( t \geq 0 \) and \( x \in I, h(\phi(t,x)) = \psi(t,h(x)) \).

**Exercise 1.1.** Show that this holds if and only if for all \( t \geq 0 \) and \( y \in J, h^{-1}(\psi(t,y)) = \phi(t,h^{-1}(y)) \).

The exercise shows that the definition is symmetric on interchange of the
equations.

2 Main result.

**Theorem 2.1** If \(-\infty < a < b < \infty, -\infty < \tilde{a} < \tilde{b} < \infty, f(a) = f(b) =
g(\tilde{a}) = g(\tilde{b}) = 0, f > 0 \) on \( ]a,b[ \), and \( g > 0 \) on \( ]\tilde{a},\tilde{b}[ \), then equation (1.1) on
\( ]a,b[ \) is topologically conjugate to (1.2) on \( ]\tilde{a},\tilde{b}[ \).
• The analogous assertion with $f$ and $g$ negative is proved in the same way. Also intervals of rightward motion bounded by equilibria are conjugate to intervals of leftward motion bounded by equilibria. The continuous map $y = h(x)$ in that case will be strictly monotone decreasing.

• In general one cannot do better than this with regards to differentiability. The proof constructs $h$ that is differentiable on $]a,b[$ with differentiable inverse. Typically at least one of $h$ and $h^{-1}$ is not differentiable at $a$.

**Example 2.1** Indeed if $0 < \lambda_2 < \lambda_1$, then the example in §4.2 of Hirsch-Smale-Devaney shows that the unique topological conjugacy $h$ between $x' = \lambda_1 x$ and $x' = \lambda_2 x$ on $[0,\infty[$ satisfying $h(1) = 1$ is given by

$$h(x) = x^{\lambda_2/\lambda_1}.$$  

This is not differentiable at $x = 0$.

**Proof of Theorem.** The idea is to let the dynamics define the conjugacy. From the fundamental theorem of the phase line, we know that the intervals $I$ and $J$ are invariant. As $t \to \infty$ orbits approach the right hand equilibria and $t \to -\infty$ the left hand.

Pick a point $x_1 \in ]a,b[$ and $y_1 \in ]\tilde{a},\tilde{b}[$. We show that there is a unique conjugacy $h(x)$ with $h(x_1) = y_1$.

For any $x \in ]a,b[$ there is a unique time $-\infty < t(x) < \infty$ so that

$$\phi(t(x), x_1) = x.$$  

The function $t(x)$ is a strictly increasing function of $x$. That $t(x)$ is differentiable follows from the differentiability of $\phi$ with $\phi_x > 0$, together with the implicit function theorem.

If there were a conjugacy with $h(x_1) = y_1$ it would satisfy

$$h(x) = h(\phi(t(x), x_1)) = \psi(t(x), h(x_1)) = \psi(t(x), y_1).$$  

(2.1)

This shows that there is only one possibility.

Furthermore, since $t(x) \to \infty$ as $x \to b$ it follows that $h$ has a continuous extension to $x = b$ by setting $h(b) = \tilde{b}$. Similarly defining $h(a) = \tilde{a}$ yields a continuous strictly increasing map of $[a,b]$ onto $[\tilde{a},\tilde{b}]$.

The inverse of a strictly increasing continuous function is also a continuous strictly increasing function proving the invertibility of $h$.  

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It remains to show that
\[ h(\phi(t, x)) = \psi(t, h(x)). \] (2.2)

If \( x \) is an endpoint this is immediate.
Equation (2.1) implies that for all \(-\infty < t < \infty\)
\[ h(\phi(t, x_1)) = \psi(t, h(x_1)) \] (2.3)
If \( x \) is not an endpoint, write \( x = \phi(t(x), x_1) \) so \( \phi(t, x) = \phi(t + t(x), x_1) \).
Compute using (2.1)-(2.3),
\[ h(\phi(t, x)) = h(\phi(t+t(x), x_1)) = \psi(t+t(x), y_1)) = \psi(t, \psi(t(x), y_1)) = \psi(t, h(x)). \]
completing the proof that \( h \) is a conjugacy. \(\square\)

**Exercise 2.1** With \( \lambda_j \) and equations as in the Example 2.1, find the unique conjugacy \( g : [0, \infty[ \to [0, \infty[ \) that satisfies \( g(1) = 2 \).

One can glue conjugacies on adjacent intervals to yield conjugacies on the union. In this way the result just proved makes the definition of equivalence in the handout on Dynamics in Dimension 1 even more natural.