Independent Eigenvector Theorem

Theorem. *If A is an $N \times N$ complex matrix with N distinct eigenvalues, then any set of N corresponding eigenvectors form a basis for $\mathbb{C}^N$.*

Proof. It is sufficient to prove that the set of eigenvectors is linearly independent.

Denote by $z_1, z_2, \ldots z_N$ the eigenvalues and corresponding eigenvectors $V_1, \ldots, V_N$.

Since each $V_j \neq 0$, any dependent subset of the $\{V_j\}$ must contain at least two eigenvectors.

If there is such a dependent pair. Choose it. If not ask if there is a dependent set of three $V's$. If yes, choose it. If not ask if there is a dependent set of 4 $V's$ and so on.

If the set of eigenvectors were dependent we would arrive at a set of j eigenvectors which is dependent and so that no subset of $j-1$ eigenvectors is dependent. Renumbering we may assume the dependent set is $V_1, \ldots, V_j$.

Since they are dependent there are constants a_k not all zero so that

$$a_1 V_1 \ + \ \cdots \ + \ a_j V_j \ = \ 0\,.$$

Since not all the a_k vanish, renumbering we may suppose that $a_j \neq 0$. Then with $b_k = -a_k/a_j$ one has

$$V_j \ = \ b_1 V_1 \ + \ \cdots \ + \ b_{j-1} V_{j-1}\,. \tag{1}$$

Since $V_j \neq 0$ the b_k are not all equal to zero.

Multiply (1) by the matrix A to find

$$z_j V_j \ = \ b_1 z_1 V_1 \ + \ \cdots \ + \ b_{j-1} z_{j-1} V_{j-1}\,. \tag{2}$$

Multiply (1) by z_j to find

$$z_j V_j \ = \ b_1 z_j V_1 \ + \ \cdots \ + \ b_{j-1} z_j V_{j-1}\,. \tag{3}$$

Subract (2) from (3) to find

$$0 \ = \ b_1(z_j - z_1) V_1 \ + \ \cdots \ + \ b_{j-1}(z_j - z_{j-1}) V_{j-1}\,. \qquad (3)$$

Since the b_j are not all zero and the differences of the eigenvalues are all not zero, this shows that $V_1, \ldots, V_{j-1}$ are dependent. By construction, $V_1, \ldots, V_{j-1}$ is independent and we arrive at a contradiction.

It follows that the eigenvectors are independent. ∎