

Ellipse Axes. Aspect ratio, and, Direction of Rotation for Planar Centers

This handout concerns 2×2 constant coefficient real homogeneous linear systems $X' = AX$ in the case that A has a pair of complex conjugate eigenvalues $a \pm ib$, $b \neq 0$. The orbits are elliptical if $a = 0$ while in the general case, $e^{-at}X(t)$ is elliptical. The latter curves are the solutions of the equation

$$X' = (A - aI)X, \quad a = \frac{\operatorname{tr} A}{2}.$$

For either elliptical or spiral orbits we associate this modified equation that has elliptical orbits. The new coefficient matrix $A - (\operatorname{tr} A/2)I$ has trace equal to zero and negative determinant. These two conditions characterize the matrices with a pair of non zero complex conjugate purely imaginary eigenvalues. Those are the matrices for which the phase portrait is a center. We show how to compute the axes of the ellipse, the eccentricity of the ellipse, and the direction of rotation, clockwise or counterclockwise. Oddly, these issues are not usually discussed in texts.

§1. Direction of rotation.

To determine the direction of rotation it suffices to find the direction of the rotation on the positive x -axis. If the flow is upward (resp. downward) then the swirl is counterclockwise (resp. clockwise). For a matrix A which has no real eigenvalues, the direction of swirl is counterclockwise if and only if the y coordinate of the product

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

is positive, if and only if $a_{21} > 0$. The swirl is counterclockwise if and only if $a_{21} > 0$.

Freeing this computation from the choice $(1, 0)$ introduces an interesting real quadratic form. For any real $X \neq 0$, the vectors X and AX cannot be parallel. Otherwise, X would be an eigenvector with real eigenvalue. Therefore the quadratic form

$$Q_1(X) := \det[X, AX] = AX \cdot X^\perp, \quad X^\perp := (-x_2, x_1).$$

is nonzero for all real $X \neq 0$. Therefore Q_1 is either always positive or always negative on $\mathbb{R}^2 \setminus 0$. The preceding criterion shows that the sign of $Q_1((1, 0))$ and therefore the sign of $Q_1(X)$ determines the direction of rotation.

This result can also be understood considering the angle θ in polar coordinates. On any curve $x(t)$ that does not touch the origin,

$$\frac{d\theta(x_1(t), x_2(t))}{dt} = \frac{\partial \theta}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial \theta}{\partial x_2} \frac{dx_2}{dt} = \frac{x_1 x_2' - x_2 x_1'}{x_1^2 + x_2^2}, \quad (1.1)$$

the last equality involving the partials of θ .

Exercise 1.1. Compute the partial derivatives of θ by differentiating the identity

$$\|x\|e^{i\theta(x)} = x$$

with respect to x_1 and x_2 .

Formula (1.1) applied to solutions of $X' = AX$ yields

$$\frac{d\theta}{dt} = \frac{x_1 x_2' - x_2 x_1'}{x_1^2 + x_2^2} = \frac{AX \cdot X^\perp}{|X|^2} = \frac{Q_1(X)}{|X|^2}.$$

Therefore Q_1 has a geometric interpretation,

$$Q_1(X) = |X|^2 d\theta/dt. \tag{1}$$

The criterion for counterclockwise rotation is $d\theta/dt > 0$.

Algorithm I. $Q_1(X)$ is a definite quadratic form and the direction of rotation is counterclockwise iff $Q_1 > 0$ on $\mathbb{R}^2 \setminus 0$, and clockwise iff $Q_1 < 0$.

Replacing A by $A - \alpha I$ multiplies the solutions of the differential equation by $e^{-\alpha t}$ and does not change $d\theta/dt$ since $X \cdot X^\perp = 0$. In particular,

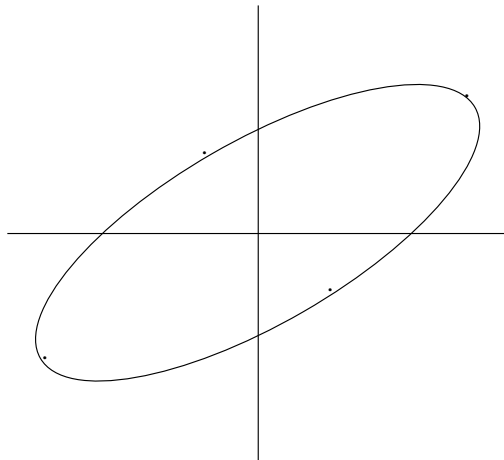
$$Q_1(X) = AX \cdot X^\perp = (A - (\text{tr } A/2)I)X \cdot X^\perp.$$

§2. Ellipse axis directions.

Define a second quadratic form associated to the matrix A with complex conjugate eigenvalues. First replace A by $A - (\text{tr } A/2)I$ that has elliptical orbits and purely imaginary eigenvalues. Define a second quadratic form

$$Q_2(W) := (A - (\text{tr } A/2)I)W \cdot W.$$

In this case one must subtract $(\text{tr } A/2)I$, it changes the form. *The points where $(A - (\text{tr } A/2)I)X \perp X$ are the points where $Q_2 = 0$. For elliptical orbits, these are the directions of the principal axes.*



The plane is divided into four quadrants by these directions. In each quadrant $Q_2(W)$ has a fixed sign and changes sign exactly when v is along one of the principal axes.

Algorithm II. For a center, the axes of the ellipse are the nonzero vectors W so that $Q_2(W) = 0$. When the ellipse is noncircular this gives a pair of perpendicular lines that are the direction of the principal axes of the ellipse.

Examples. i. For

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}, \quad Q_2(X) = AX \cdot X \quad AX = (x_2, -4x_2), \quad Q(X) = -3x_1x_2.$$

The equation of the $0 = Q_2(W) = 4w_1w_2$. axes is $x_1x_2 = 0$ so the axes are the usual euclidean axes.

ii. For

$$A = \begin{pmatrix} 3 & 5 \\ -2 & -2 \end{pmatrix},$$

$\text{tr } A = 3 - 2 = 1$, so

$$A - (\text{tr } A/2)I = \begin{pmatrix} 2.5 & 5 \\ -2 & -2.5 \end{pmatrix}$$

is the trace free matrix whose motion is a center. The equation determining the axes is

$$0 = Q_2(W) = 2.5 w_1^2 + 3w_1w_2 - 2.5 w_2^2.$$

Any nonzero solution must have $w_1 \neq 0$. Dividing by w_1 shows that solutions are all multiples of $W = (1, y)$ where,

$$2.5 + 3y - 2.5 y^2 = 0, \quad \text{equivalently,} \quad 5y^2 - 6y - 5 = 0. \quad (2)$$

The roots are

$$\frac{6 \pm \sqrt{6^2 - 4(-5)(5)}}{10} = \frac{6 \pm \sqrt{136}}{10} = \frac{3 \pm \sqrt{34}}{5}.$$

The two roots yield the directions $(1, y_1)$ and $(1, y_2)$ of the two axes of the ellipse. The orthogonality of the directions is equivalent to the fact that the product of the roots is equal to -1 . This follows from (2) since in the quadratic equation for y , the constant term and the coefficient of y^2 differ by a factor -1 .

iii. In the degenerate case where there are more than two directions for which $Q_2 = 0$, one has a quadratic equation with more than two roots so the quadratic form vanishes identically and Q_2 is identically equal to zero. In this case the orbits are circular.

§3. Major and minor axes.

The differential equation with coefficient $\tilde{A} := A - (\text{tr } A/2)I$ has elliptical orbits. Compute orthogonal unit vectors W_j along the axes of the associated ellipse using Algorithm II. Denote by Y the coordinates with respect to the basis W_j ,

$$X = y_1W_1 + y_2W_2. \quad (3.1)$$

Introduce the matrix T^{-1} whose first column is W_1 and second column is W_2 so

$$T^{-1} := \begin{pmatrix} W_{1,1} & W_{2,1} \\ W_{1,2} & W_{2,2} \end{pmatrix}.$$

Equation (3.1) asserts that

$$X = T^{-1}Y, \quad \text{equivalently} \quad Y = TX. \quad (3.2)$$

From the definition of T^{-1} one has

$$T^{-1}(1, 0) = W_1, \quad T^{-1}(0, 1) = W_2, \quad \text{so,} \quad TW_1 = (1, 0), \quad TW_2 = (0, 1). \quad (3.3)$$

Exercise 3.1 i. If the columns of a 2×2 matrix M form an orthonormal basis of \mathbb{R}^2 show that $M^\dagger M = I$ where \dagger denotes transpose. **ii.** Use this to show that $(T^{-1})^\dagger T^{-1} = I$. Derive from this that $TT^\dagger = T^\dagger T = I$.

$X(t)$ satisfies $X' = \tilde{A}X$ if and only if $Y = TX$ satisfies $Y' = T\tilde{A}T^{-1}Y$. Define

$$\alpha := \tilde{A}W_2 \cdot W_1, \quad \beta := \tilde{A}W_1 \cdot W_2. \quad (3.4)$$

Then,

$$T\tilde{A}T^{-1} = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}. \quad (3.5)$$

Proof. The first column of $T\tilde{A}T^{-1}$ is equal to

$$T\tilde{A}T^{-1}(1, 0) = T\tilde{A}W_1.$$

Expand $\tilde{A}W_1$ with respect to the orthonormal basis of W 's to find

$$\tilde{A}W_1 = (\tilde{A}W_1 \cdot W_1)W_1 + (\tilde{A}W_1 \cdot W_2)W_2 = \beta W_2.$$

Therefore

$$T\tilde{A}W_1 = \beta TW_2 = (0, \beta)$$

verifying the first column of (3.5).

Exercise 3.2. Finish the proof of (3.5) by verifying the second column of the identity. ■

The formulas show that α and β are real. The determinant of A is negative and equal to the determinant of $T\tilde{A}T^{-1}$ so $\alpha\beta < 0$ proving that α and β have opposite sign.

The differential equation in Y coordinates is

$$y_1' = \alpha y_2, \quad y_2' = \beta y_1.$$

Multiply the first equation by βy_1 , the the second by αy_2 , and subtract to find

$$0 = \beta y_1 y_1' - \alpha y_2 y_2' = \frac{1}{2} \frac{d(\beta y_1^2 - \alpha y_2^2)}{dt}.$$

The orbits have equation $\beta y_1^2 - \alpha y_2^2 = \text{constant}$. The phase portrait is a center iff these curves are ellipses iff α and β have opposite signs. In that case, the orbits are similar (in the sense of Euclidean geometry) to the ellipse $|\beta|y_1^2 + |\alpha|y_2^2 = 1$. The axes are along the y -coordinate axes. The axis along the y_1 -axis (resp. y_2 -axis) has length $|\beta|^{-1/2}$ (resp. $|\alpha|^{-1/2}$).

Algorithm III. Suppose that W_j are orthogonal unit vectors along the axis directions found in Algorithm II and α and β are computed from formula (2). Then the elliptical orbits are similar to the ellipse with axis along W_1 of length $|\beta|^{-1/2}$ and axis along W_2 of length $|\alpha|^{-1/2}$. In particular the ratio of the longer axis to the shorter axis (the aspect ratio) is equal to

$$\frac{\max\{|\alpha|^{1/2}, |\beta|^{1/2}\}}{\min\{|\alpha|^{1/2}, |\beta|^{1/2}\}}.$$

Examples. i. Choose $W_1 = (1, 0)$, $W_2 = (0, 1)$ unit vectors along the axes computed in §2. Then

$$\begin{aligned}\alpha &= \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -4 \\ \beta &= \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1.\end{aligned}$$

The ellipses have axes along the x_1 and x_2 axis. The major axis is along x_1 and is longer by a factor 2 than the minor axis.

ii. Choose,

$$U_1 = \left(1, \frac{3 + \sqrt{34}}{5}\right), \quad W_1 = \frac{U_1}{\|U_1\|}, \quad U_2 = \left(1, \frac{3 - \sqrt{34}}{5}\right), \quad W_2 = \frac{U_2}{\|U_2\|}.$$

Then

$$\begin{aligned}\alpha &= \frac{1}{\|U_1\|\|U_2\|} \begin{pmatrix} 2.5 & 5 \\ -2 & -2.5 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{3+\sqrt{34}}{5} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \frac{3-\sqrt{34}}{5} \end{pmatrix}, \\ \beta &= \frac{1}{\|U_1\|\|U_2\|} \begin{pmatrix} 2.5 & 5 \\ -2 & -2.5 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{3-\sqrt{34}}{5} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \frac{3+\sqrt{34}}{5} \end{pmatrix},\end{aligned}$$

The ellipses are similar to the ellipse with axes along W_1 and W_2 of lengths $|\beta|^{-1/2}$ and $|\alpha|^{-1/2}$ respectively.

Exercise. If the axes are not along the x -axes, then the equation for $x = x_1/x_2$ in algorithm II has the form $x^2 + ax - 1 = 0$. Show that it is impossible to find an example where the roots are integers. **Hint.** The sum of the roots is equal to $-a$ and the discriminant must be a perfect square.

This may explain why you don't find algorithms II, III in textbooks.