

Remarks on existence and uniqueness

§1. Continuous dependence.

The last assertion of our existence and uniqueness assertion was that

$$\|X(t) - \tilde{X}(t)\| \leq e^{\Lambda|t-t_0|} \|X_0 - \tilde{X}_0\|.$$

The proof (when $t_0 = 0$) begins with the two equations

$$X(t) = X_0 + \int_0^t F(s, X(s)) ds, \quad \text{and} \quad \tilde{X}(t) = \tilde{X}_0 + \int_0^t F(s, \tilde{X}(s)) ds.$$

Subtracting yields

$$X(t) - \tilde{X}(t) = X_0 - \tilde{X}_0 + \int_0^t F(s, X(s)) - F(s, \tilde{X}(s)) ds.$$

Therefore

$$\|X(t) - \tilde{X}(t)\| \leq \|X_0 - \tilde{X}_0\| + \int_0^t \|F(s, X(s)) - F(s, \tilde{X}(s))\| ds$$

Use the estimate

$$\|F(s, X(s)) - F(s, \tilde{X}(s))\| \leq \Lambda \|X(s) - \tilde{X}(s)\|$$

to find

$$\|X(t) - \tilde{X}(t)\| \leq \|X_0 - \tilde{X}_0\| + \int_0^t \Lambda \|X(s) - \tilde{X}(s)\| ds.$$

The desired estimate then follows by applying Gronwall's inequality from page 393 of the text with

$$u(t) := \|X(t) - \tilde{X}(t)\|, \quad C := \|X_0 - \tilde{X}_0\|, \quad \text{and} \quad K := \Lambda.$$

§2. Locally defined F .

Consider the initial value problem

$$X' = F(t, X), \quad X(t_0) = X_0, \tag{1}$$

with

$$F \in C^1(\Omega)$$

where Ω is an open subset of \mathbb{R}^{1+n} containing (t_0, X_0) . We show how Picard's existence theorem suffices to prove local existence and uniqueness for this seemingly more general problem.

Since Ω is open one can choose $r > 0$ so small that

$$B := \left\{ (t, X) \in \mathbb{R}^{1+n} : |t - t_0| + \|X - X_0\| < r \right\} \subset \Omega.$$

Choose $\chi(t, x)$ to be a C^∞ function which is identically equal to 1 on a neighborhood of (t_0, X_0) and so that χ vanishes outside of B .

Define $G(t, X)$ to be the function which is equal to χF in B and vanishes outside of B . Then $G \in C^1$ and its derivatives vanish outside a compact set so are uniformly bounded.

It follows from the existence theorem proved in class that the initial value problem

$$X' = G(t, X), \quad X(t_0) = X_0 \tag{2}$$

has a global solution $X \in C^1(\mathbb{R})$.

Since G vanishes outside a bounded set and is continuous,

$$K := \sup_{\mathbb{R}^{1+d}} \|G(t, X)\| < \infty.$$

Then for the solution of (2),

$$\|X(t) - X_0\| = \left\| \int_0^t X'(s) ds \right\| \leq \int_0^t \|G(s, X(s))\| ds \leq \int_0^t K ds \leq K|t - t_0|.$$

Therefore there is a $\delta > 0$ so that for $t_0 \leq t \leq t_0 + \delta$ the solution $X(t)$ lies in the set where $\chi = 1$. On that set, $F = G$ so that for $t_0 \leq t \leq t_0 + \delta$, X satisfies (1). In this way we have constructed a solution of (1) on the time interval $[t_0, t_0 + \delta]$. This proves local existence.

For uniqueness suppose that X and \tilde{X} are solutions of (1) defined for $t_0 \leq t \leq T$.

By continuity of X and \tilde{X} there is a $\delta > 0$ so that for $t_0 \leq t \leq t_0 + \delta$, $(t, X(t))$ and $(t, \tilde{X}(t))$ belong to the set where $\chi = 1$.

Therefore, for $t_0 \leq t \leq t_0 + \delta$ both X and \tilde{X} are solutions of the initial value problem (2).

By the known uniqueness of solutions of the initial value problem (2) we conclude that $X = \tilde{X}$ for $t_0 \leq t \leq t_0 + \delta$.

It is not hard (but not entirely trivial either) to show by a "marching" scheme that $X = \tilde{X}$ on the entire interval $t_0 \leq t \leq T$.

Interested readers can ask me for details of the last argument in office hours.