Math 558 Fall 2006 Prof. J.Rauch

Theorems on Gradient Systems

The systems have the form

$$X' = -\mathrm{grad}\, V(X)\,.$$

On solutions one has

$$V(X(t))' = -\,\|\mathrm{grad}\, V(X(t))\|^2\,.$$

Equilibria are the critical points of V. If X^* is an equilibrium, the linearization is

$$Y' = -D^2V(X^*)\,Y\,,$$

where D^2V denotes the $N \times N$ symmetric matrix of second derivative,

$$D^2V(X^*) := \frac{\partial^2 V}{\partial x_i \partial x_j}(X^*)\,.$$

This matrix is called the *Hessian*. *If the eigenvalues are all strictly positive (they must be real by symmetry) then, X^* is a sink. In particular, it is asymptotically stable.* Strictly positive eigenvalues is the usual second derivative sufficient condition for a strict local minimum.

The function

$$L(X) := V(X) - V(X^*)\,,$$

decreases on orbits and vanishes at X^*. *L is a Lyapunov function when (and only when) X^* is a strict local minimum of V. It is a strict Lyapunov function when in addition X^* is an isolated critical point point of V. Lyapunov's Theorem implies that in the first case, X^* is stable, and in the second it is asymptotically stable.*

Examples. 1. When $D^2V(X^*)$ is strictly positive, it follows that X^* is an isolated critical point which is a strict minimum. **2.** The example (with 26 continuous derivatives)

$$V(X) := |X|^{27}\Big(1 + \sin^2(1/|X|)\Big)\,,$$

has a strict local minimum but there are circles $|X| = r_k$ with $r_k \to 0$ consisting of equilibria, so X^* is NOT asymptotically stable. This shows that the isolation hypothesis in the above result is needed.

For any α the sets $\{V(X) \le \alpha\}$ are positively invariant. Lasalle's Principle yields the following important estimate for the basin of attraction.

Theorem. *If X^* is a strict local minimum of V with $V(X^*) < \alpha$ and if the connected component of X^* in $\{X \,:\, V(X) \le \alpha\}$ is compact and contains no equilibria other than X^*, then that component is contained in the basin of attraction of X^*.*

This proves the intuitive result that orbits starting in the bowl defined by the local minimum X^* descend the bowl to X^*. The complete phase portrait of one such example is on page 206 of the text.