

Linearization at an Equilibrium II. Perturbation Theory Approach

The first derivation of linearization is by approximating $f(X)$ for X approximately equal to an equilibrium \underline{X} using Taylor's Theorem

$$f(X + \delta X) \approx f'(\underline{X})\delta X, \quad f'(\underline{X}) := \begin{bmatrix} \frac{\partial f_1(\underline{X})}{\partial X_1} & \cdots & \frac{\partial f_1(\underline{X})}{\partial X_n} \\ \frac{\partial f_2(\underline{X})}{\partial X_1} & \cdots & \frac{\partial f_2(\underline{X})}{\partial X_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n(\underline{X})}{\partial X_1} & \cdots & \frac{\partial f_n(\underline{X})}{\partial X_n} \end{bmatrix}.$$

This approximation drops terms higher order in δX .

The same result is obtained here using Perturbation Theory. The theorem on differentiable dependence then gives a rigorous justification of the linearization method.

Introduce a small parameter ϵ writing the initial value as $\underline{X} + \epsilon Z$,

$$X'(t, \epsilon) = f(X(t, \epsilon)), \quad X(0, \epsilon) = \underline{X} + \epsilon Z. \quad (1)$$

Setting $\epsilon = 0$ yields the equation for the unperturbed solution $X_0(t) := X(t, 0)$,

$$X'_0(t) = f(X_0(t)), \quad X_0(0) = \underline{X}. \quad (2)$$

The unperturbed solution is the equilibrium,

$$X_0(t) = \underline{X}. \quad (3)$$

Differentiate (1) with respect to ϵ to find for all t, ϵ ,

$$\frac{\partial}{\partial t} \frac{\partial X}{\partial \epsilon} = f'(X) \frac{\partial X}{\partial \epsilon}. \quad (4)$$

Differentiate the initial condition, (2), with respect to ϵ to find,

$$\frac{\partial X(0, \epsilon)}{\partial \epsilon} = Z. \quad (5)$$

Define

$$X_1(t) := \frac{\partial X(t, 0)}{\partial \epsilon}, \quad \text{so } X(t, \epsilon) \approx X_0(t) + \epsilon X_1(t) \quad (6)$$

is the first order Taylor approximation.

Set $\epsilon = 0$ in (4),(5) using (3), to find the linearized initial value problem determining $X_1(t)$,

$$X'_1(t) = f'(\underline{X}) X_1(t), \quad X_1(0) = Z. \quad (7)$$

The approximation to the error $\delta X := \epsilon X_1(t)$ satisfies

$$\delta X' = f'(\underline{X}) \delta X. \quad (8)$$

This is the same result obtained by ignoring terms in δX of order higher than 1 science text style derivation given earlier.