

Conjugating the Tent and Logistic Maps

In the Hirsch, Smale and Devaney the chaotic character of the logistic map is proved by constructing a semiconjugation of the tent and logistic maps. The exercise at the end of this note shows that there is a conjugation by an even cleverer trigonometric map. There is something deeper going on that the trigonometric identity does not reveal.

- All that matters is that the map goes steeply to the top then steeply down.
- A conjugation if it exists, is uniquely and easily determined.

1 n -cycles.

1.1 n -cycles of the Tent Map.

1. The tent map f has two fixed points. One is at 0. See figure 15.10 of HSD. The slope of f at the fixed points is 2 so they are repellers.
2. $f \circ f$ has 2^2 fixed points. Two are fixed points of f . Two additional fixed points of $f \circ f$ form a 2-cycle of f . All are repellers.

Exercise 1.1 *Counting from the left show that the fixed points of f are in first and third position among the fixed points of $f \circ f$.*

3. $f \circ f \circ f$ has 2^3 fixed points. All repellers. Two are fixed points of f . The remaining six form two three cycles.

Exercise 1.2 *Show that the fixed points of f are in first and sixth position among the fixed points of $f \circ f \circ f$. Hint.* Figure 15.10 of HSD.

4. f^4 has 2^4 4-cycles. 2^2 are fixed points of f^2 . The remaining 12 form three four cycles.
5. The graph of f^n consists of 2^{n-1} tents of width 2^{1-n} . Each tent has two n -cycles one on the left side and one on the right. See figure 15.10 page 340 of HSD.
6. The distance between adjacent n -cycles is no larger than 2^{2-n} .

Adjacent n -cycles are connected by the graph rising to the top and bouncing back to $y = x$ or sinking to the floor and bouncing back. The slopes of the segments are equal to 2^{n+1} . This yields the following lower bound.

7. For any compact subinterval $[a, b] \subset]0, 1[$ there is a constant $c(a, b)$ independent of n to that the distance between adjacent n -cycles in $[a, b]$ satisfies

$$c 2^{-n} < \text{distance} < 2^{-n}.$$

1.2 n -cycles of the Logistic Map.

There are analogous results for the logistic map $g(x) = 4x(1 - x)$ from $[0, 1]$ to itself.

1. The logistic map g has two fixed points. One is at 0. Both are repellers. See figure 15.7 of HSD.
 2. $g \circ g$ has 2^2 fixed points. Two are fixed points of g and the other two form a 2-cycle of g . All unstable.
 3. g^3 has 2^3 fixed points. Two are fixed points. The remaining six form two three cycles.
 4. g^4 has 2^4 fixed points. 2^2 are fixed points of g^2 . The remaining 12 form three four cycles.
 5. The graph of g^n consists of 2^{n-1} fingers with width $\leq 1/2^{n-1}$. Each tent has two n -cycles.
- The next two assertions are clear from figure 15.7 page 337 of HSD. The first uses the fact that the fingers are no wider than $C 2^{-n}$.
6. The distance between adjacent n -cycles is no larger than $2/2^{n-1}$.
 7. For any compact subinterval $[a, b] \subset]0, 1[$ there is a constant $c(a, b)$ independent of n to that the distance between adjacent n -cycles in $[a, b]$ is no smaller than $c 2^{-n}$.

The proof is as for the tent map. The graph must reach the top or the bottom and return. It covers a vertical distance that is bounded below. For the one tenth of the path that is nearest the top or bottom the slope is bounded above by $C 2^n$. Together this yields the lower bound.

For both the logistic and tent maps the set of all cycles is dense in $[0, 1]$. This is one of the three defining characteristics of chaotic maps.

2 Conjugations and n cycles

Definition A homeomorphism h is a **conjugation between maps f and g** when $f = h^{-1} \circ g \circ h$.

1. Then x is an n -cycle of f if and only if $h(x)$ is an n -cycle of g .
2. h is a homeomorphism of $[0, 1]$ when and only when h is a surjective strictly monotone map. It is increasing when $h(0) = 0$ and decreasing in case $h(0) = 1$. Since the tent map and the logistic map have only one endpoint that is a fixed point and that is 0, any conjugation must map 0 to itself. Therefore any conjugation is strictly increasing with $h(1) = 1$.
3. It follows that for each n any conjugation of the tent map to the logistic map must map the n -cycles of the tent map to the n -cycles of the logistic map preserving the order. Thus any conjugation is uniquely determined on the set of all cycles.
4. Since the set of all cycles of the tent map are dense there is at most one conjugation.

Theorem 2.1 *The tent map is topologically conjugate to the logistic map.*

Proof. Step I. If there is a conjugacy then its values at the endpoints and all n -cycles is uniquely determined.

The conjugation h must take the nonzero fixed point of f to the nonzero fixed point of g .

The fixed points of $f \circ f$ must be mapped to the equal number of fixed points of $g \circ g$ in increasing order. The mapping on the fixed points of f has already been assigned. It is not obvious that the fixed points of $f \circ f$ can be mapped in increasing order while respecting the previous assignment. For example, if there were two fixed points of $f \circ f$ between the fixed points of f and only one fixed point of $g \circ g$ between the fixed points of g .

A necessary and sufficient condition for being able to assign values consistently is the following. For each n and $1 \leq m \leq 2^n$ denote by $x_{m,n}$ the m^{th} n -cycle of f^n counting from the left

and $y_{m,n}$ the analogous cycles for g^n . The nasc is that that for each $k < n$ and $1 \leq j \leq 2^k$ $x_{j,k} \in]x_{m,n}, x_{m+1,n}]$ if and only if $y_{j,k} \in]y_{m,n}, y_{m+1,n}]$.

To convince you that this is satisfied consider the j^{th} fixed point of f^k . It is in one of the intervals of length 2^{-k} . Call that interval J . All the intervals of size 2^{-n} lying to the left of the interval contain fixed points of f^n that lie to the left. To decide the ranking one need only consider the intervals of width 2^{-n} starting at the left endpoint of J .

For intervals of length 2^{-n} consider them as open on the left and closed on the right. Then the fixed point lies in exactly one of the intervals of size 2^{-n} whose union is J . Call that small interval K . Of the two fixed points of f^n in K our j^{th} is either to the left or the right of center. Exactly this picture is reproduced for the logistic map and shows that the ranking of the j^{th} fixed point of g^k among the fixed points of g^n is exactly the same.

In this way the value of h on all n -cycles is uniquely determined. This determines h on a dense subset.

Step II. *It suffices to show that the mapping defined on the cycles extends to a continuous function on $[0, 1]$.*

Since the restriction of h to the set of cycles is strictly increasing it follows that any continuous extension is nondecreasing. To show that it is strictly increasing suppose that $x_1 < x_2$. Choose two cycles $\tilde{x}_1 < \tilde{x}_2$ between x_1 and x_2 . Then $h(x_1) \leq h(\tilde{x}_1) < h(\tilde{x}_2) \leq h(x_2)$ proving that h is strictly monotone.

Step III. Construction of a continuous extension.

To extend to the open subinterval $]0, 1[$ it suffices to extend to each compact subinterval $[a, b] \subset]0, 1[$. To extend to $[a, b]$ it is sufficient (and necessary) to show that if $z_n \neq w_n$ are sequences of cycles in $[a, b]$ with $z_n - w_n \rightarrow 0$ then $h(z_n) - h(w_n) \rightarrow 0$.

To verify the sufficient condition, denote by $r(n)$ and $s(n)$ be the degrees of z_n and w_n . Renaming z_n and w_n we may suppose that $r(n) \geq s(n)$. Since $0 \neq z_n - w_n \rightarrow 0$, it follows that $r(n) \rightarrow \infty$. In addition the nearest $r(n)$ cycle \tilde{z}_n to w_n and further from z_n is at most 2^{-n} further from z_n so $z_n - \tilde{z}_n \rightarrow 0$. Suppose that the ranks of z_n and \tilde{z}_n among fixed points of f^n differ by $j_n > 0$. Then $z_n - \tilde{z}_n \rightarrow 0$ if and only if $j_n/2^n \rightarrow 0$.

The images $h(z_n)$ and $h(\tilde{z}_n)$ are fixed points of g^n and $h(\tilde{z}_n)$ is the j_n^{th} neighbor of $h(z_n)$ among such cycles. It follows that $|h(z_n) - h(\tilde{z}_n)| \leq C j_n 2^{-n} \rightarrow 0$. Since $h(w_n)$ lies between z_n and \tilde{z}_n it follows that $|h(z_n) - h(w_n)| \leq |h(z_n) - h(\tilde{z}_n)| \rightarrow 0$. This verifies the sufficient condition for a continuous extension. Thus h extends to a continuous function on $]0, 1[$.

To complete the proof it suffices to show that $\lim_{x \rightarrow 0} h(x) = 0$ and $\lim_{x \rightarrow 1} h(x) = 1$. The largest n -cycle x_n of the tent map and its image y_n both converge to 1. Since h is monotone with values in $[0, 1]$ it follows that as $x \rightarrow 1$, $h(x) \rightarrow 1$. Considering the smallest n -cycle in $]0, 1[$ proves that $\lim_{x \rightarrow 0} h(x) = 0$. Therefore h extended so that $h(0) = 0$ and $h(1) = 1$ is continuous on $[0, 1]$. ■

Exercise 2.1 *For the tent map f and logistic map g and homeomorphism*

$$k(x) := \frac{2}{\pi} \arcsin(\sqrt{x})$$

show that $f = k^{-1} \circ g \circ k$. **Discussion.** The conjugation constructed above has this simple formula. For more general maps for which the construction works there will be no trigonometric miracle.