Phase Planes for Two by Two Linear Systems

1 Introduction.

Consider linear constant coefficient systems
\[ X' = AX, \quad (1.1) \]
where \( A \) is a real matrix with distinct non zero eigenvalues. This is an open dense subset of the set of four dimensional set of \( 2 \times 2 \) real matrices. The complementary set is a closed set of dimension three, hence consists of rare occurences defined by,
\[ \det A = 0, \quad \text{or} \quad \text{discriminant} = 0. \]
The condition of distinct nonzero eigenvalues is stable under small perturbations of \( A \) which is another way of saying that the set is open. In class we proved the the complementary set is closed and of four dimensional volume equal to zero. The systems studied here are generic.

2 Change of variables

A sub theme in these computations is the behavior of a linear constant coefficient system
\[ X' = AX \]
when one makes a linear change of variable \( \alpha = TX \) where \( T \) is an invertible matrix. This computation is valid for general \( N \times N \) systems in which case the matrix \( T \) is also \( N \times N \). It is found in §3.4 of HSD.

Writing \( X(t) = T^{-1}\alpha(t) \), the differential equation for the \( \alpha(t) \) is
\[ (T^{-1}\alpha)' = AT^{-1}\alpha. \]
The left hand side is equal to \( T^{-1}\alpha' \) so multiplying on the left by \( T \) yields
\[ \alpha' = TAT^{-1}\alpha. \quad (2.1) \]
The new variables satisfy a system of the same form with the matrix \( A \) changed by a similarity to \( TAT^{-1} \).

If you understand the \( TAT^{-1} \) differential equation then you understand the original problem. The strategy is to make \( TAT^{-1} \) as simple as possible so the transformed problem is easier.

3 Saddles. Real eigenvalues of opposite signs.

If \( A \) has real eigenvalues of opposite signs \( \lambda_- < 0 < \lambda_+ \) then choosing nonzero real eigenvectors \( v_- \) and \( v_+ \) the general solution is
\[ X(t) = \alpha_- e^{\lambda_- t} v_- + \alpha_+ e^{\lambda_+ t} v_+. \]
The line \( \mathbb{R}v_+ \) is invariant under the flow. The flow at time \( t \) simply multiplies by \( e^{\lambda_+ t} \) so points move outward exponentially. As \( t \to -\infty \) they converge exponentially to 0. This line is called the \textit{unstable manifold}. 

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The line $\mathbb{R}v_-$ is also invariant and the flow is multiplication by $e^{\lambda_- t}$ so contracts exponentially toward the origin. As $t \to -\infty$ the flow expands exponentially toward infinity. This line is called the stable manifold.

Since $Av_+ = \lambda_+ v_+$, the vector field on $\mathbb{R}v_+$ points parallel to the line and away from the origin. The length is proportional to the distance from the origin.

The vector fields along $\mathbb{R}v_-$ is similar except pointing inward.

As $t \to -\infty$ the integral curves are asymptotic to the line $\mathbb{R}v_-$. As $t \to \infty$ the integral curves are asymptotic to the line $\mathbb{R}v_+$.

An integral curve off the stable and unstable manifolds comes in from the direction of $\mathbb{R}v_-$ turns and leaves approaching $\mathbb{R}v_+$ as $t \to \infty$. They have an aspect that is roughly hyperbolic.

To see that they are NOT hyperbolas when the modulus of the $\lambda$'s are not equal note that the approach to the asymptote corresponding to the eigenvalue with smaller modulus is more rapid than the approach to the other since the decay of the negligible term is more rapid.

This is easily seen in the example

$$X' = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} X.$$

The general solution is

$$x_1 = c_1 e^{\lambda_- t}, \quad x_2 = c_2 e^{\lambda_+ t}.$$

Therefore on orbits one has

$$|x_1|^{\lambda_+} / |x_2|^{\lambda_-} = \text{independent of time.}$$

Since $\lambda_- < 0$, $|x_1|^{\lambda_+} / |x_2|^{\lambda_-} = |x_1|^{\lambda_+} |x_2|^{\lambda_-}$. Therefore the continuous function

$$\varphi(x_1, x_2) := |x_1|^{\lambda_+} |x_2|^{\lambda_-}$$

is constant on orbits. When $|\lambda_+| = |\lambda_-|$ this shows that the orbits are hyperbolas. And when the lambdas are not of equal magnitude the orbit is not a hyperbola.

We next show by a change of basis that there is always a continuous quantity like (3.1) that is constant on orbits. Introduce the components $v_+ = (v_{+1}, v_{+2})$ and the matrix

$$T := \begin{pmatrix} v_{-1} & v_{+1} \\ v_{-2} & v_{+2} \end{pmatrix}^{-1}, \quad T^{-1} := \begin{pmatrix} v_{-1} & v_{+1} \\ v_{-2} & v_{+2} \end{pmatrix}.$$

Then

$$T^{-1}(1, 0) = v_-, \quad T^{-1}(0, 1) = v_+, \quad Tv_- = (1, 0), \quad Tv_+ = (0, 1),$$

where the last two follow from the first two upon multiplying by $T$.

For $X \in \mathbb{R}^2$ denote by $(\alpha_-, \alpha_+) := \alpha$ the coordinates of $X$ in the basis $v_-, v_+$ that is

$$X = \alpha_1 v_- + \alpha_+ v_+ = \begin{pmatrix} v_{-1} & v_{+1} \\ v_{-2} & v_{+2} \end{pmatrix} \alpha.$$
so,
\[ X = T^{-1} \alpha, \quad \alpha = TX. \]

Compute the equation satisfied by \( \alpha \) using (2.1). Equation (3.2) shows that
\[ T A T^{-1}(1,0) = T A v_+ = T \lambda_- v_- = \lambda_-(1,0). \]

Similarly \( T A T^{-1}(0,1) = \lambda_+(0,1) \). Therefore,
\[ T A T^{-1} = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}, \]
and the differential equation satisfied by the coordinates \( \alpha \) is
\[ \alpha' = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} \alpha, \quad \alpha' = \lambda_- \alpha_- \quad \alpha' = \lambda_+ \alpha_. \]

This is the example analysed earlier so we know that \( \varphi(\alpha) = \varphi(TX) \) is a continuous conserved quantity. Also called an integral of motion.

The figure on page 92 of Brauer and Nohel and page 41 of Hirsch-Smale-Devaney are exactly for the exceptional case of \( \lambda_- = -\lambda_+ \) with hyperbolic orbits. The reader is encouraged to plot (using pplane in Matlab) an example where this condition is violated to see the symmetry breaking which is typical.

The expression \( \varphi(\alpha) \) is constant on orbits. It is called a conserved quantity or an integral of motion. In terms of the original coordinates and the matrix elements \( T_{ij} \) one has
\[ \alpha_- = T_{11} x_1 + T_{12} x_2, \quad \alpha_+ = T_{21} x_1 + T_{22} x_2, \]
so a conserved quantity is
\[ \varphi(\alpha) = |\alpha_1|^{\lambda_+} |\alpha_2|^{\lambda_-} = |T_{11} x_1 + T_{12} x_2|^{\lambda_+} |T_{21} x_1 + T_{22} x_2|^{\lambda_-}. \]

**Summary.** There are two invariant lines, in the directions of the eigenvectors. The flow is outward (resp. inward) in the direction of the eigenvector with positive (resp. negative) eigenvalues.

The equilibrium is unstable. The other integral curves are asymptotic to the to these two lines and resemble hyperbolas, but lacking their symmetry and quadric equation (unless \( \lambda_- = -\lambda_+ \)).

There is a nontrivial continuous integral of motion.

**4 Improper nodes. Distinct real roots of the same sign.**

Consider the case of positive roots \( 0 < \lambda_1 < \lambda_2 \). The general solution is
\[ c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2. \]

The lines in the directions of \( v_j \) are invariant and the flow is outward on both. The invariant lines correspond to one or the other of \( c_1, c_2 \) vanishing. The expansion on the \( v_2 \) line is more rapid so that orbits tend in the limit \( t \to \infty \) to be nearly parallel to \( v_2 \). For \( t \to -\infty \) the \( e^{\lambda_1 t} \) term is dominant and the orbits approach the origin tangent to the \( v_1 \) line.

Introducing the basis \( v_1, v_2 \) and corresponding coordinates \( \alpha_1, \alpha_2 \) and matrices \( T^{-1}, T \) as in the preceding section one has
\[ T A T^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \alpha = (c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}). \]
The quantity
\[ \left| \frac{\alpha_2}{\alpha_1} \right|^{\lambda_1} = \frac{c_1^{\lambda_1}}{c_2^{\lambda_2}} e^{\lambda_1 \lambda_2 t} = \frac{c_1^{\lambda_1}}{c_2^{\lambda_2}} \]
is constant on orbits. Since \( \lambda_2 > 0 \) this is discontinuous along the line \( \alpha_1 = 0 \).

The orbits in \( \alpha \) coordinates are as in figure 2.9 of Brauer and Nohel and figure 3.3.b of Hirsch-Smale-Devaney. The orbits in \( X \) space are their image by the linear transformation \( T^{-1} \). Qualitatively they look like figure 3.7 in HSD with arrows reversed.

Orbits move away from the origin growing infinitely large. The origin is a source or repellor.

**Proposition 4.1** If the eigenvalues of \( A \) both have strictly positive (resp. negative) real part, then the only continuous conserved quantities are constants.

**Proof.** We treat the case of positive real part. Suppose that \( \varphi \) is a continuous conserved quantity. If \( P \) is any point, Denote by \( X(t) \) the orbit with \( X(0) = P \). Then for all \( t \),
\[ \varphi(P) = \varphi(X(0)) = \varphi(X(t)) \, . \]

As \( t \to -\infty \), \( X(t) \to 0 \). By continuity of \( \varphi \) at the origin,
\[ \varphi(0) = \lim_{t \to -\infty} \varphi(X(t)) = \varphi \left( \lim_{t \to -\infty} X(t) \right) = \varphi(P) \, . \]

Therefore \( \varphi \) is constant.

The case of distinct negative real eigenvalues is analogous to the case of positive distinct eigenvalues generating a sink which is an attractor. The negative case behaves as the positive case with time reversed.

**Summary.** There are two invariant lines. For positive real part, the flow is outgoing on both. The equilibrium is unstable. The line corresponding to the larger eigenvalue dominates for \( t \) large positive, while the other dominates for \( t \to -\infty \). There are no nonconstant continuous conserved quantities.

5 Centers and spirals. Complex conjugate eigenvalues.

Solutions are generated by
\[ e^{\lambda t} v, \quad e^{\overline{\lambda} t} \overline{v} \, . \]
Equivalently, by the real and imaginary parts of \( e^{\lambda t} v \). Write
\[ \lambda = a + ib, \quad v = r + is, \]
in terms of their real and imaginary parts. Then
\[ e^{\lambda t} v = e^{at} e^{ibt} v \, . \]

The term \( e^{ibt} v \) is of constant magnitude so there is exponential growth (resp. exponential decay) when \( a > 0 \) (resp. \( a < 0 \)). The analysis is most conveniently done by considering first the case \( a = 0 \).
5.1 Centers. Eigenvalues $0 \neq \pm bi$.

The expression becomes,

\[(\cos bt + i \sin bt)(r + is) = (r \cos bt - s \sin bt) + i(r \sin bt + s \cos bt).\]

Real solutions are generated by the real and imaginary parts,

\[r \cos bt - s \sin bt, \quad r \sin bt + s \cos bt.\]

Introduce the basis $r, s$ and corresponding coordinates and the notation $T$ following the case of saddles,

\[X = \alpha_1 r + \alpha_2 s = \begin{pmatrix} r_1 & s_1 \\ r_2 & s_2 \end{pmatrix} \alpha := T^{-1} \alpha, \quad \alpha = TX.\]

Therefore

\[T^{-1}(1,0) = r, \quad T^{-1}(0,1) = s, \quad (1,0) = Tr, \quad (0,1) = Ts. \tag{5.1}\]

Compute

\[TAT^{-1}(1,0) = TA r. \tag{5.2}\]

Use

\[A(r + is) = ib((r + is), \quad \text{equivalently,} \quad Ar + iAs = -bs + ibr.\]

Taking the real and imaginary parts yield

\[Ar = -bs, \quad \text{and,} \quad As = br.\]

Continuing with (5.2) and using (5.1) yields

\[TAT^{-1}(1,0) = TA r = T(-bs) = -b(0,1).\]

Similarly $TAT^{-1}(0,1) = b(1,0)$ so

\[TAT^{-1} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\]

and the equations for the $\alpha$ coordinates are

\[\alpha' = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \alpha, \quad \alpha'_1 = b \alpha_2, \quad \alpha'_2 = -b \alpha_1. \tag{5.3}\]

**Exercise 5.1.** Verify the identity. Also verify $TAT^{-1}(0,1) = \lambda_{+}(0,1)$ in the section on saddles and $TAT^{-1}(0,1) = b(1,0)$ in the section on centers.

Multiply the first equation in (5.3) by $\alpha_1$, the second by $\alpha_2$, and add to find that

\[\alpha_1 \alpha'_1 + \alpha_2 \alpha'_2 = \alpha_1 b \alpha_2 - \alpha_2 b \alpha_1 = 0.\]

This proves that $\alpha_1^2 + \alpha_2^2$ is constant on orbits so the orbits are circles in the $\alpha$ coordinates. See figure 2.11 in Brauer and Nohel and 3.4 in Hirsch-Smale-Devaney. The conserved quantity in $X$ coordinates is computed using

\[\alpha_1 = T_{11} x_1 + T_{12} x_2, \quad \alpha_2 = T_{21} x_1 + T_{22} x_2,\]
to be

\[(T_{11}x_1 + T_{12}x_2)^2 + (T_{21}x_1 + T_{22}x_2)^2.\]

Its level sets are bounded conic sections, hence ellipses. The orbits in the \(X\) coordinates are ellipses. In a separate handout we address the question of computing the principal axes, eccentricity, and direction of rotation for the elliptical orbits.

**Summary.** For nonzero purely imaginary eigenvalues, the orbits are ellipses. There is a quadratic conserved quantity. The origin is a stable equilibrium. Under small real perturbations of \(A\), the eigenvalues will typically leave the imaginary axis remaining a complex conjugate pair.

### 5.2 Spirals. Eigenvalues \(a \pm ib, a \neq 0 \neq b\).

The solutions are exactly as in the preceding section just multiplied by \(e^{at}\). For \(a > 0\) the orbits are ellipses amplified by an exponentially growing factor. They spiral out. For \(a < 0\) they spiral in. The orbits in \(\alpha\) coordinates are given in figure 2.10 of Brauer and Nohel and fig 3.5 of Hirsch-Smale-Devaney.

**Summary** For \(a > 0\) the orbits are elliptical spirals growing exponentially called a spiral source. For \(a < 0\) they are elliptical spirals shrinking exponentially, a spiral sink. There are no nonconstant continuous integrals of motion.