Phase Planes for Two by Two Linear Systems

1 The structurely stable cases.

Consider linear constant coefficient systems

$$X' \;=\; AX\,, \tag{1.1}$$

where A is a real matrix with distinct non zero eigenvalues. This is an open dense subset of the set of four dimensional set of 2×2 real matrices. The complementary set is a closed set of dimension three, hence consists of rare occurences defined by,

$$\det A \;=\; 0, \qquad \text{or,} \qquad \text{discrimnant} \;=\; 0.$$

The condition of distinct nonzero eigenvalues is stable under small perturbations of A which is another way of saying that the set is open.

2 Saddles. Real eigenvalues of opposite signs.

If A has real eigenvalues of opposite signs $\lambda_- < 0 < \lambda_+$ then choosing nonzero real eigenvectors v_- and v_+ the general solution is

$$X \;=\; \alpha_- e^{\lambda_1 t}\, v_- \;+\; \alpha_+ e^{\lambda_+ t}\, v_+\,.$$

The line $\mathbb{R}v_+$ is invariant under the flow. The flow at time t simply multiplies by $e^{\lambda_+ t}$ so points move outward exponentially. As $t \to -\infty$ they converge exponentially to 0. This line is called the *unstable manifold*.

The line $\mathbb{R}v_-$ is also invariant and the flow is multiplication by $e^{\lambda_- t}$ so contracts exponentially toward the origin. As $t \to -\infty$ the flow expands exponentially toward infity. This llne is called the stable manifold.

Since $Av_+ = \lambda_+ v_+$, the vector field on $\mathbb{R}v_+$ points parallel to the line and away from the origin. The length is proportional to the distance from the origin.

The vector fields along $\mathbb{R}v_-$ is similar except pointing inward.

As $t \to \infty$ the $c_+ e^{\lambda_+ t}$ term is dominant and integral curves are asymptotic to the line $\mathbb{R}v_+$.

As $t \to -\infty$ integral curves are asymptotic to the line $\mathbb{R}v_-$.

An integral curve off the stable and unstable manifolds comes in from the direction of $\mathbb{R}v_-$ turns and leaves approaching $\mathbb{R}v_+$ as $t \to \infty$. They have an aspect that is roughly hyperbolic.

To see that they are NOT hyperbolas when the modulus of the λ's are not equal note that the approach to the asymptote correponding to the eigenvalue with smaller modulus is more rapid than the approach to the other since the decay of the negligible term is more rapid.

Another way to see this is to make a change of basis as follows. Introduce the components $(v_{+,1}\,,\, v_{+,2})$ and the matrix

$$\begin{pmatrix} v_{-,1} & v_{+,1} \\ v_{-,2} & v_{+,2} \end{pmatrix}$$

whose first (resp. second) column is v_- (resp. v_+). The columns being linearly independent this is a nonsingular matrix so we can define

$$T \ :=\ \begin{pmatrix} v_{-,1} & v_{+,1} \\ v_{-,2} & v_{+,2} \end{pmatrix}^{-1} .$$

Then

$$T^{-1}(1,0)\ =\ v_-, \qquad T^{-1}(0,1)\ =\ v_+ . \tag{2.1}$$

Equivalently,

$$Tv_-\ =\ (1,0), \qquad Tv_+\ =\ (0,1) . \tag{2.2}$$

The numbers $(\alpha_-, \alpha_+) := \alpha$ are the coordinates of X in the basis v_-, v_+ that is

$$X\ =\ \alpha_1\, v_-\ +\ \alpha_+\, v_+\ =\ \begin{pmatrix} v_{-,1} & v_{+,1} \\ v_{-,2} & v_{+,2} \end{pmatrix} \alpha,$$

so,

$$X = T^{-1}\alpha, \qquad \alpha = TX .$$

The differential equation is equivalent to

$$\big(T^{-1}\alpha\big)'\ =\ A\,T^{-1}\alpha .$$

The left hand side is equal to $T^{-1}\alpha'$ so multiplying on the left by T yields

$$\alpha'\ =\ T\,A\,T^{-1}\alpha .$$

Using first (2.1) then (2.2) shows that

$$T\,A\,T^{-1}(1,0)\ =\ T\,A\,v_-\ =\ T\,\lambda_- v_-\ =\ \lambda_- v_- .$$

Similarly $TAT^{-1}(0,1) = \lambda_+(0,1)$. Therefore,

$$TAT^{-1}\ =\ \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix},$$

and the differential equation satisfied by the coordinates α is

$$\alpha_-' = \lambda_- \alpha_- \,, \qquad \alpha_+' = \lambda_+ \alpha_+ \,,$$

In matrix form,

$$\alpha'\ =\ \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} \alpha .$$

The α_+ (resp. α_-) axis is the unstable (resp. stable) manifold. The solution curves are

$$(\alpha_- \,,\, \alpha_+)\ =\ \big(c_- e^{\lambda_- t} \,,\, c_+ e^{\lambda_+ t}\big) .$$

Each quadrant in α space is invariant. In the quadrant of positive coordinates

$$\alpha_-^{\lambda_+} = c_-^{\lambda_+}\, e^{\lambda_-\lambda_+ t}, \qquad \alpha_+^{\lambda_-} = c_+^{\lambda_+}\, e^{\lambda_-\lambda_+ t} .$$

Dividing shows that

$$\frac{\alpha_-^{\lambda_+}}{\alpha_+^{\lambda_-}}\ =\ \text{constant} .$$

Hyperbolas are conic sections. They have equations which are polynomial of degree two. The hyperbolas asymptotic to the α-axes have equation $\alpha_-\alpha_+ =$ constant, so when $\lambda_- \neq -\lambda_+$ the integral curve is not a hyperbola. The integral curve in x space is the image by T^{-1} of this curve so is also not a conic section since those sections are invariant under linear maps.

The figure 2.8 on page 92 is exactly for the exceptional case of $\lambda_- = -\lambda_+$ with hyperbolic orbits. The reader is encouraged to plot using `pplane` in matlab an example where this condition is violated to see the symmetry breaking which is typical.

The expression $\alpha_-^{\lambda_+}/\alpha_+^{\lambda_-}$ is constant on orbits. It is called a *conserved quantity* or an *integral of motion*. In the other quadrants a similar computation shows that $|\alpha_-|^{\lambda_+}/|\alpha_+|^{\lambda_-}$ is constant on orbits. In terms of the original coordinates and the matrix elements T_{ij} one has

$$x_1 = T_{11}\alpha_- + T_{12}\alpha_+, \qquad x_2 = T_{21}\alpha_- + T_{22}\alpha_+,$$

so the conserved quantity is,

$$\frac{|T_{11}\alpha_- + T_{12}\alpha_+|^{\lambda_-}}{|T_{21}\alpha_- + T_{22}\alpha_+|^{\lambda_+}}\,.$$

Summary. *There are two invariant lines, in the directons of the eigenvectors. The flow is outward (resp. inward) in the direction of the eigenvector with positive (resp. negative) eigenvalues. The other integral curves are asymptotic to the to these two lines resembling hyperbolas, but lacking their symmetry and quadric equation (unless $\lambda_- = -\lambda_+$).*

3 Improper nodes. Distinct real roots of the same sign.

Consider the case of positive roots $0 < \lambda_1 < \lambda_2$. The general solution is

$$c_1\, e^{\lambda_1 t}\, v_1 \;+\; c_2\, e^{\lambda_2 t}\, v_2\,.$$

The lines in the directions of v_j are invariant and the flow is outward on both. The invariant lines correspond to one or the other of c_1, c_2 vanishing. The expansion on the v_2 line is more rapid so that orbits tend in the limit $t \to \infty$ to be nearly parallel to v_2. For $t \to -\infty$ the $e^{\lambda_1 t}$ term is dominant and the orbits approach the origin tangent to the v_1 line.

Introducing the basis v_1, v_2 and corresponding coordinates α_1, α_2 and matrices T^{-1}, T as in the preceding section one has

$$TAT^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \alpha = \big(c_1 e^{\lambda_1 t}\,,\, c_2 e^{\lambda_2 t}\big)\,.$$

The quantity

$$\frac{|\alpha_1|^{\lambda_2}}{|\alpha_1|^{\lambda_2}}$$

is constant on orbits.

The orbits in α coordinates are as in figure 2.9 of Brauer and Nohel. The orbits in X space are the image of this by the linear transformation T^{-1}.

Orbits move away from the origin growing infinitely large. The origin is a *source*.

The case of negative eigenvalues is analogous, generating a *sink*.

Summary. *There are two invariant lines. The flow is outgoing on both. The line corresponding to the larger eigenvalue dominates for t large positive, while the other dominates for $t \to -\infty$. There is a conserved quantity with transcendental formula as in the case of saddles.*

4 Centers and spirals. Complex conjugate eigenvalues.

Solutions are generated by

$$e^{\lambda t} v, \qquad e^{\overline{\lambda} t}\overline{v}\,.$$

Equivalently, by the real and imaginary parts of $e^{\lambda t}v$. Write

$$\lambda = a + ib, \qquad v = r + is,$$

in terms of their real and imaginary parts. Then

$$e^{\lambda t} v \;=\; e^{at}\, e^{ibt}\, v\,.$$

The term $e^{ibt}v$ is of constant magnitude so there is exponential growth (resp. exponential decay) when $a > 0$ (resp. $a < 0$). The analysis is most conveniently done by considering first the case $a = 0$.

4.1 Centers. Eigenvalues $0 \neq \pm bi$.

The expression becomes,

$$\big(\cos bt + i\sin bt\big)(r + is) \;=\; \big(r\,\cos bt - s\,\sin bt\big) \;+\; i\big(r\,\sin bt + s\,\cos bt\big)\,.$$

Real solutions are generated by the real and imaginary parts,

$$r\,\cos bt - s\,\sin bt\,, \qquad r\,\sin bt + s\,\cos bt\,.$$

Introduce the basis r, s and corresponding coordinates,

$$X \;=\; \alpha_1 r \;+\; \alpha_2 s \;=\; \begin{pmatrix} r_1 & s_1 \\ r_2 & s_2 \end{pmatrix} \alpha \;:=\; M^{-1}\alpha\,,$$

defining the matrix M. The general solution is

$$\alpha \;=\; \big(c_1 \cos bt + c_2 \sin bt\,,\, -c_1 \sin bt + c_2 \cos bt\big)\,.$$

The coordinates are.

$$\alpha_1 = c_1 \cos bt + c_2 \sin bt, \qquad \alpha_2 = -c_1 \sin bt + c_2 \cos bt\,.$$

By inspection,

$$\alpha_1' \;=\; b\,\alpha_2, \qquad \alpha_2' \;=\; -b\,\alpha_1, \qquad \alpha' \;=\; \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \alpha\,.$$

Multiply the first equation by α_1, the second by α_2 and add to find that

$$\alpha_1\alpha_1' \;+\; \alpha_2\alpha_2' \;=\; \alpha_1\, b\,\alpha_2 \;-\; \alpha_2\, b\,\alpha_1 \;=\; 0.$$

This proves that $\alpha_1^2 + \alpha_2^2$ is constant on orbits so the orbits are circles in the α coordinates. See figure 2.11 in Brauer and Nohel. The conserved quantity in X coordinates is computed using

$$\alpha_1 \;=\; M_{11}x_1 + M_{12}x_2\,, \qquad \alpha_2 \;=\; M_{21}x_1 + M_{22}x_2\,,$$

to be

$$\left(M_{11}x_1 + M_{12}x_2\right)^2 \;+\; \left(M_{21}x_1 + M_{22}x_2\right)^2 .$$

Its level sets are ellipses showing that the orbits in the X coordinates are ellipses.

In a separate handout we address the question of computing the prinicipal axes, excentricity, and direction of rotation for the elliptical orbits.

Summary. *For nonzero purely imaginary eigenvalues, the orbits are ellipses. There is a quadratic conserved quantity. The origin is a stable equilibrium. Under small real perturbations of A, the eigenvalues will typically leave the imaginary axis remaining a complex conjugate pair*

4.2 Spirals. Eigenvalues $a \pm ib$, $a \neq 0 \neq b$.

The solutions are exactly as in the preceding section just multiplied by e^{at}. For $a > 0$ the orbits are ellipses amplified by an exponentially growing factor. They spiral out. For $a < 0$ they spiral in. The orbits in α coordinates are given in figure 2.10 of Brauer and Nohel.

Proposition 4.1 *The only continuous conserved quantities are constants.*

Proof. Suppose that φ is a continuous conserved quantity. If P is any point, Denote by $X(t)$ the orbit with $X(0) = P$. Then

$$\varphi(P) \;=\; \varphi(X(0)) \;=\; \varphi(X(t)) .$$

As $t \to -\infty$, $X(t) \to 0$ so by continuity of φ at the origin,

$$\varphi(0) \;=\; \lim_{t\to\infty} \varphi(X(t)) \;=\; \lim_{t\to-\infty} \varphi(P) \;=\; \varphi(P),$$

proving that φ is constant. ■

Summar *For $a > 0$ the orbits are elliptical spirals growing exponentially called a spiral source. For $a < 0$ they are elliptical spirals shrinking exponenetially, a spiral sink*