Introductory lecture.

Energy law for the Klein-Gordon equation. i. For real solutions of the Klein=Gordon equation,

$$u_{tt} - \Delta u + u = 0,$$

prove the differential energy law

$$\partial_t (u_t^2 + |\nabla_x u|^2 + u^2) \ + \ \mathrm{div}(2\, u_t \nabla_x u) \ = \ 0.$$

ii. Derive from this the fact that if $u(0,x) = u_t(0,x) = 0$ for $|x| \geq R$, then $u = 0$ for $|x| \geq R + |t|$. **Hint.** In order to have integrals over finite sets it is easier to prove that u vanishes on $\{R + |t| \leq |x| \leq M - |t|\}$ with $M > R$ arbitrary.

iii. For compactly supported initial data show that

$$\int u_t^2(t,x) + |\mathrm{grad}\, u(t,x)|^2 + u^2(t,x) \ dx$$

is independent of t.

Discussion. This shows that the $L^2(\mathbb{R}^d)$ norm of u is bounded uniformly in time. For D'alembert's wave equation the L^2 norm can grow in time as the next examples show. If $f(s)$ is a smooth function on $\mathbb{R}$ which is equal to 1 for $s \leq 0$ and vanishes for s large positive, then

$$u = f(x-t) - f(x+t) \quad \text{for} \quad d = 1, \qquad \text{or} \qquad u = \frac{f(|x| - t) - f(|x| + t)}{|x|} \quad \text{for} \quad d = 3,$$

is a smooth, compactly supported solution of $u_{tt} = \Delta u$ and for t large there is a constant $c > 0$ so that

$$\int u^2(t,x) \ dx \ \geq \ c|t|^d\,.$$

This growth of the $L^2(\mathbb{R}^d)$ norm is in sharp contrast to the Klein-Gordon equation. In physical problems where D'Alembert's equation occurs, either the energy is measured by $\partial_{t,x} u$ (e.g. acoustics) or there are supplementary equations which guarantee that the L^2 growth does not occur (e.g. Maxwell's equations).

Plane waves and finite speed. The fundamental finite speed assertion for the wave equation asserts that if $u \in C^2(\mathbb{R}^{1+d})$ satisfies

$$u_{tt} = c^2 \Delta u, \qquad c > 0,$$

and,

$$u(0,x) = u_t(0,x) = 0 \quad \text{for} \quad |x| \leq R,$$

then,

$$u(t,x) = u_t(t,x) = 0, \quad \text{for} \quad |x| \leq R - c|t|.$$

Use plane waves or spherical waves to show that for any $\sigma < c$ one cannot conclude that $u = 0$ for $|x| \leq \sigma |t|$. **Discussion.** If one could so conclude it would say that signals traveled no faster than σ which would have been a stronger conclusion. Draw a sketch!

§**1.4.**

Method of Images. i. Verify that $u := e^{-|x|}$ satisfies

$$\Big(- \frac{d^2}{dx^2} + 1 \Big) u \;=\; \delta(x)\,.$$

Hint. Use the definition of distribution derivatives.

ii. Find a solution of

$$\Big(- \frac{d^2}{dx^2} + 1 \Big) u \;=\; \delta(x-a)\,.$$

iii. Suppose that $a > 0$. Construct a solution $u(x)$ continuous on $x \ge 0$ and so that

$$\Big(- \frac{d^2}{dx^2} + 1 \Big) u \;=\; \delta(x-a) \qquad u(0) = 0\,, \qquad \lim_{x\to\infty} u(x) \;=\; 0\,.$$

Discussion. It is not hard to show using the maximum principal, that there is only one such solution u.

iv. Suppose that $a > 0$. Construct a solution $u(x)$ continuous on $x \ge 0$ and so that

$$\Big(- \frac{d^2}{dx^2} + 1 \Big) u \;=\; \delta(x-a) \qquad \frac{du(0)}{dx} = 0\,, \qquad \lim_{x\to\infty} u(x) \;=\; 0\,.$$

Hint. Even functions satisfy the Neumann condition at $x = 0$ while odd functions satisfy the Dirichlet condition.

§**1.5.**

Fish location and Snell's law. Standing on shore a fisherman with eyes at height h above the water looks into the water at a fish which is swimming at depth d below a point L units of distance away. The line of sight passes A units above the fish making the fish look to be less deep than it really is. A spear fisherman who does not correct for this effect will throw the spear A units above the fish. Compute a formula for A in terms of h, d, L.