Outline

1. Friedrichs’ dream, boundary value problems without reference to type.
2. Rectangular domains of computation leads to problems in domains with corners.
3. Standard PDE theory
   - Characteristic polynomial.
   - Elliptic operators
   - Hyperbolic operators
   - Plane waves
4. A motivating example from bounded operators
5. A little real analysis
   - Principal of dense convergence
   - Mollifiers $j_\epsilon * \rightarrow I$
6. Differential operators and adjoints
   - Integration by parts formula
   - Definition of symmetric postive systems
7. Estimates and existence theorem for symmetric postive systems on $\mathbb{R}^d$
8. Examples
   - $d = 1$
     - failure of ellipticity
     - singular solutions
   - Transport for $d > 1$
     - failure of ellipticity
     - singular solutions
   - Maxwell’s equations
     - laplace transformed equations
     - the exponential trick
9. Friedrichs’ Lemma
10. Applications
    - Integration by parts formula
    - Uniqueness
    - Weak = Strong

11. Symmetric hyperbolic systems
ODE on $\mathbb{C}^N$ decreasing euclidean norm
Symmetric positive operators generate contractive evolutions on $L^2(\mathbb{R}^d)$.
Exponential trick reduces to a symmetric positive system on $\mathbb{R}^{1+d}$
$A_0 \partial_t$ with strictly positive $A_0$

Electromagnetism example: calcite

12. Linearized compressible inviscid flow
Symmetric form not invariant by $L \mapsto M(x)L$ nor by $u \mapsto M(x)\tilde{u}$

13. Determining the number of boundary conditions needed for a boundary value problem
Constant transport in a half space
The complementarity condition, ODE case (see online handout)
Spectral condition for symmetric positives
Tangential Fourier transform argument, Lopatinski complementarity.

14. $H^1(\Omega)$ is dense in $\mathcal{H}_L$

15. Boundary traces
First trace theorem, $u \in \mathcal{H}_L \Rightarrow u \big|_{\partial \Omega} \in H^{-1/2}(\partial \Omega)$
$v \in L^2(I; H^s)$ and $v' \in L^2(I; H^{-s})$ imply $v \in C(I; L^2)$
Second trace theorem
$u, v \in \mathcal{H}_L, \times \mathcal{H}_L^\perp \Rightarrow \langle \sum A_j \nu_j u, v \rangle \big|_{\partial \Omega} \in \text{Lip}(\Omega)'$
Greens’ identity holds
Connection with compensated compactness

16. Adjoint boundary space
Equivalent definitions of weak solutions
Equivalent definitions of strong solutions

17. Weak and strong solutions, existence and uniqueness
Inequality implying existence of weak solutions
Inequality implying uniqueness of strong solutions

18. Proof of weak=strong
Follows Lax-Phillips 1960. Tangential smoothing + Friedrichs Lemma

19. Positive boundary conditions (Online handout posted)
positive, strictly positive, conservative
Maximal positive boundary spaces
Algebra of positive boundary conditions

20 The fundamental existence and uniqueness Theorems
A priori estimates for positive boundary conditions


22. Differentiability
For $\Omega = \mathbb{R}^d$
For bounded domains with noncharacteristic boundary
  Normal form at boundary. Change $x$, change $u$, multiply.
  Tangential derivative.
  Then noncharacteristic argument.

**23.** Generalization to boundaries characteristic of constant multiplicity
  Example of Maxwell equations.

**24.** $\Omega \in \mathbb{R}^2$ equal to a quadrant
  Coupled transport examples
  Elliptic version

**25.** Elliptic strictly dissipative generators are OK at corners.