Encounters with Partial Differential Equations

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Every analyst has the Laplacian, the Wave Operator, and the Fourier Transform as close friends. This short course (three or four meetings of three hours) consists of three encounters, one with each of these friends. It is intended for people with some background in analysis, including the elements of Lebesgue integration, complex analysis in one variable, and basic functional analysis. A brief exposure to the three friends is desirable. Even experts will find much that they have not seen.

The aim rather than to follow a shortest straight line path to a certain goal is to reveal the many interconnections between subjects. To explore the fascinating byways that one avoids in the straight line paths. The interconnectedness reveal a a deeper and more complete portrait of the fundamental objects. Symmetry is the core idea in the differential equations encounters.

**Encounter I. Meet the Laplacian.** Centers on spherical symmetry and the Mean Value Property.

- Stationary Dirichlet integral/surface area.
- Rotational invariance of $\Delta$ and harmonic functions. Invariant harmonic functions.
- Haar measure on the orthogonal group.
- Symmetry derivation of MVP.
- The laplacian as infinitesimal MVP.
- Orthogonally invariant polynomials. Second proof of MVP.
- Gravitational field of charge distribution. At large distances.
- Newton’s theorem on the gravitational field of a spherical shell. Proof by symmetry.
- Euclid’s symmetry proof of equal angles in isosceles triangles.
- Proof of Newton’s Theorem by MVP.
- The five point Laplacian.
- Discrete heat flow.
- Random walk and the discrete Dirichlet problem.
• Random walk and the Dirichlet problem.

**Encounter II. Fourier analysis derived from complex analysis.**

Fourier series preceded complex analysis. An independent derivation of the basic results is given using purely complex techniques is presented. Arguments based on approximate delta functions are replaced systematically by the Cauchy Integral Formula. Includes Shannon’s Sampling Theorem. Online notes available.

**Encounter III. Meet the Wave Operator**

• D’Alembert’s method in 1-d.
• Radial solutions in $3 - d$. Loss of derivatives.
• No loss in $L^2$ Sobolev spaces.
• Ill posedness in $L^p$ for $p \neq 2$, $d > 1$.

The last topics introduce some asymptotic methods in a very concrete and physically important setting.

• High frequency asymptotics and the rectilinear propagation of light.
• Reflection.
• Refraction.
• Total reflection.