Convergence of series of means

After class on mercoledì 25 marzo, I was asked about the convergence of the series expansion of the mean of $u$ over balls of radius $r$ as a function of $r$.

A short and somewhat unsatisfying answer is that the integral over the ball of radius $r$ can be computed from the Taylor series by term by term integration since the series is uniformly convergent. The result is a power series in $r^2$ by inspection. Convergent by the preceding argument as soon as $u$ is analytic in $|x| < r + \varepsilon$.

Since the derivatives of an analytic function grow like $n!$ in order for the series to converge the constants in the expansion $\sum c_k \Delta^k u(0)$ must have factorials in them. The answer is unsatisfactory as one does not see the factorials. We compute exactly the coefficients $c_k$.

The Taylor series of $u$ has terms

$$\frac{\partial^\alpha u(0)}{\alpha!} x^\alpha.$$

We compute the coefficient of $\Delta^n u(0) r^{2n}$ which comes from derivatives of order $2n$. Consider the term

$$\frac{\partial^{2n} u(0)}{(2n)!} x_1^{2n}.$$

A scaling argument shows that the average over the ball of radius $r$ is equal to

$$c_n \frac{\partial^{2n} u(0)}{(2n)!} r^{2n}, \quad c_n = \frac{\int_{B_1(0)} x_1^{2n} dx}{|B_1(0)|} < 1.$$

(1)

Recovering an invariant polynomial from its values on the $x_1$ axis shows that the term of order $2n$ is equal to

$$\frac{c_n \Delta^n u(0) r^{2n}}{(2n)!}$$

with $c_n$ from (1), since the $\partial^{2n} u(0)$ term from this expression is correct. The convergence of the series follows.