TD on the Laplacian

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A function $u$ on $\mathbb{R}^d$ is **invariant under the orthogonal group** when for all $R \in SO(d)$, $u \circ R = u$. The value of such a function depends only on $|x|$. We call such functions **radial**.

The **Dirichlet integral** on the open set $\Omega$ is denoted

$$ J(u) := \int_{\Omega} |\nabla u|^2 \, dx, \quad u \in C_0^\infty(\mathbb{R}^d). $$

The Laplacian is characterized by,

$$ \frac{dJ(u + \epsilon \phi)}{d\epsilon} = -2 \int \Delta u \, \phi \, dx, \quad \phi \in C_0^\infty(\Omega). $$

1. Use this characterization to prove that for radial functions

$$ \Delta u = \frac{1}{r^{d-1}} \left( r^{d-1} u_r \right)_r, \quad r \neq 0. $$

**Hint.** Write $J$ in polar coordinates.

Denote by $\langle \cdot, \cdot \rangle$ the bilinear pairing of $\mathbb{C}^d \times \mathbb{C}^d$ defined by

$$ \langle \zeta, x \rangle := \sum \zeta_i x_i. $$

A quadratic form $q$ on $\mathbb{R}^d$ is associated with a unique symmetric matrix $B = B^t$ by $q(v) = \langle Bv, v \rangle$.

2. Prove that if $q$ is an $SO(d)$ invariant quadratic form then $q$ is a scalar multiple of $\langle v, v \rangle$.

A differential operator $P(\partial)$ with constant coefficients is **invariant** under orthogonal transformations iff for all $R \in SO(d)$ and $u \in C^\infty(\mathbb{R}^d)$

$$ P \left( u \circ R \right) = (Pu) \circ R. $$

For a constant coefficient partial differential operator replacing $\partial$ by $i\xi$ yields a polynomial $P(\xi)$ called the **symbol**. Then

$$ P(\partial) e^{i\langle \xi, x \rangle} = P(\xi) e^{i\langle \xi, x \rangle}. $$
3. Show that if \( P(\partial) \) is a constant coefficient scalar partial differential operator invariant under orthogonal transformations, then its symbol \( P(\xi) \) is an invariant polynomial. \(^1\) Conclude that if \( P \) is homogeneous of degree 2 then it is a scalar multiple of the Laplacian.

If \( p(x) \) is a polynomial it is uniquely expressed as a sum of homogeneous terms,

\[ p = p_0 + p_1 + \cdots + p_m. \]

4. Show that \( p \) is orthogonally invariant if and only if each \( p_j \) is.

5. Show that there are no invariant polynomial of degree 1. Of degree 3. Of any odd degree.

6. Show that the invariant polynomials of degree \( 2k \) are equal to \( c(x_1^2 + \cdots + x_d^2)^k \).

7. Show that every homogeneous invariant scalar partial differential with constant coefficients is of even order and is equal to \( c\Delta^k \) for some \( k \).

8. Denote by \( A(u, r) \) average of \( u \) over the ball of radius \( r \) centered at the origin. Show that there are constant coefficient homogeneous scalar partial differential operator \( P_k(\partial) \) of degree \( k \) so that

\[ A(u, r) = u(0) + \frac{1}{d+2} \Delta u(0)|x|^2 + (P_4(\partial)u(0))|x|^4 + (P_6(\partial)u(0))|x|^6 + \cdots. \]

9. Show that each operator \( P_k \) is invariant under orthogonal transformations. Conclude that \( P_{2k} = c_k \Delta^k \) for some constant \( c_k \).

**Discussion.** This proves that if \( u \) is harmonic, then the Taylor expansion of \( A(u, r) - u(0) \) as a function of \( r \) vanishes identically. This is the infinitesimal mean value property. Since harmonic functions are real analytic the Taylor series converges to \( A(u, r) - u(0) \) and one concludes that \( A(u, r) = u(0) \), the **Mean Value Property**. In my talk I presented a shorter proof using the fact that \( A(u, r) \) is a radial harmonic function. This is so since \( A \) is the average over \( R \in SO \) of \( u \circ R \). It did not use real analyticity.

\(^1\)The converse is proved using the Fourier Transform.