Automorphisms of Modular Lie Algebras

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Abstract: We give a short argument that certain modular Lie algebras have surprisingly large automorphism groups.

1. Introduction and statement of results

A classical Lie algebra is one that has a Chevalley basis associated with an irreducible root system. If \( L \) is a classical Lie algebra over a field \( K \) of characteristic 0 then \( L \) has the following properties: (1) The automorphism group of \( L \) contains the Chevalley group associated with the root system as a normal subgroup with torsion quotient group; (2) \( L \) is simple. It has been known for some time that these properties do not always hold when \( K \) has positive characteristic, even when (2) is relaxed to condition (2') \( L \) is quasi-simple, (that is, \( L/Z(L) \) is simple, where \( Z(L) \) is the center of \( L \)), but the proofs have involved explicit computations with elements of the algebras. See [Stein] (whose introduction surveys the early results in this area), [Hog].

Our first result is an easy demonstration of the instances of failure for (1) or (2') by use of graph automorphisms for certain Dynkin diagrams; see (2.4), (3.2) and Table 1. Only characteristics 2 and 3 are involved here.

We also determine the automorphism groups of algebras of the form \( L/Z \), where \( Z \) is a central ideal of \( L \) and \( L \) is one of the above classical quasisimple Lie algebras failing to satisfy (1) or (2'); see (3.8) and (3.9). As far as we know, existing literature deals only with the cases \( Z = Z(L) \) and full arguments are not published. Our proofs use the classification of simple algebraic groups with elementary arguments from the theories of Lie algebras and finite groups, plus a few fairly well-known facts about finite subgroups of algebraic groups. Our arguments are relatively noncomputational, more nearly self-contained and shorter than earlier treatments. To keep this article brief, we do not consider the nonexceptional cases, which are dealt with in [Stein] and earlier literature, though our methods would work. We also give easy proofs of some results of [Hiss] on the action of a Chevalley group on its classical Lie algebra. It is possible that our graph automorphism technique is useful on modules other than the adjoint module.

2. Preliminary Results

2.1 Definition We call a Lie algebra of classical type if it has a Chevalley basis over the field \( K \).
Thus, the automorphism group of a classical type Lie algebra contains the Chevalley group of that type.

2.2 Lemma  Let $\sigma$ be an automorphism of an algebra. Suppose that $C$ is the fixed point subalgebra and $\phi(t) \in K[t]$. Then $N = \text{Im}(\phi(\sigma))$ is stable under multiplication by $C$. In particular, $C \cap N$ is an ideal of $C$.

**Proof**  Straightforward.

The following is an immediate consequence.

2.3 Corollary  Let $A$ be an algebra over the field $K$, and let $\sigma$ be an automorphism of $A$ with fixed point subalgebra $C$. Suppose that $\sigma$ has minimal polynomial $(1-t)^k$, and let $N = \text{Im}(1-\sigma)^{k-1}$.

(i) If $C \neq N$ then $C$ is not simple.

(ii) If $N \not\subseteq Z(C)$, $Z(C)$ is an ideal and $C \neq N + Z(C)$, then $C$ is not quasi-simple.

2.4 Corollary  The following Lie algebras are not quasi-simple:

Types $B_n(K), C_n(K), F_4(K)$ when $\text{char}(K) = 2$.

Type $G_2(K)$ when $\text{char}(K) = 3$.

**Proof**  Apply the previous result to the standard graph automorphisms of $D_{n+1}, A_{2n-1}, E_6$, and $D_4$.

3. The Main Results

To obtain our main results, we apply the observations above to the following situation:

3.1 Definition  A special quadruple is a 4-tuple $(L, \sigma, M, K)$ where $L$ is a classical Lie algebra over $K$, $K$ is a field of characteristic $p > 0$, $\sigma$ is a standard graph automorphism of $L$, $\sigma$ has order $p$, and $M$ is the fixed point subgroup for the action of $\sigma$ on the Chevalley group associated to $L$; $M$ is itself a Chevalley group, associated with the fixed points of $\sigma$ on the root lattice; thus, $M$ is a group of automorphisms of the fixed point subalgebra of $\sigma$.

Notice that we are identifying $\sigma$ with an automorphism of the root system, say, $\Phi$ and that there is understood to be a set $\Pi$ of fundamental roots and a Chevalley basis $\mathcal{B} = \{h_\alpha, e_\beta \mid \alpha \in \Pi, \beta \in \Phi\}$ permuted by $\sigma$ as $\sigma$ permutes the subscripts.

3.2 Lemma  Let $(L, \sigma, M, K)$ be a special quadruple, $C := \text{Ker}(1-\sigma)$ and let $N = \text{Im}(1-\sigma)^{p-1}$. Then $M$ acts on the Lie algebra $C/N$. Furthermore $C$ contains a subalgebra $S$ of classical type such that $C = N + S$ and $S \cap N \subseteq Z(S)$ where $Z(S)$ is the center of $S$. If $L$ is of type $D_n$ or $E_6$ then the root system associated to $S$ is irreducible. For any central subalgebra $Z$ of $S$, $M$ acts faithfully on $S/Z$.

3.3 Definition  We call $S$ a covering subalgebra for the quadruple $(L, \sigma, M, K)$.

3.4 Notation  Define $Q := C/N$. Also, when $K$ is algebraically closed, let $A := \text{Aut}(S/Z(S))^0$; this is an algebraic group which contains an isomorphic copy of $M$ as
an algebraic subgroup. Let \( R \) be the Chevalley group associated to \( S; \) by definition, it is a subgroup of \( \text{Aut}(S) \).

**Proof** [of (3.2)] It is clear that \( N \subseteq C \), so that \( N \) is an ideal in \( C \) and \( M \subseteq \text{Aut}(C) \) acts on \( C/N \). The action is faithful, as we can see by passing to the algebraic closure where \( M \) becomes a direct product of simple groups. For convenience, let \( H := (h_\alpha | \alpha \in \Pi) \) and let \( E_0 \) be the subspace spanned by \( \{e_\beta | \beta \text{ is fixed by } \sigma \} \).

Let \( S \) be the subalgebra generated by \( E_0 \). Then \( S = S \cap H \oplus E_0 \subseteq C \), and \( S \cap N \subseteq \mathbb{Z}(S) \cap H \). In fact, \( \{\sum_{\alpha \in \mathcal{O}} h_\alpha | \mathcal{O} \text{ is a regular } \sigma \text{-orbit on } \Pi\} \) is a basis for \( S \cap N \).

For each \( \sigma \)-orbit \( \mathcal{O} \) on \( \Pi \), let \( \bar{\mathcal{O}} \) be the smallest connected subset of nodes containing \( \mathcal{O} \). Also let \( \gamma_\mathcal{O} = \sum_{\beta \in \bar{\mathcal{O}}} \beta \). Then \( \{\gamma_\mathcal{O} | \mathcal{O} \text{ is a } \sigma \text{-orbit on } \Pi\} \) is a fundamental system of roots for the fixed points of \( \sigma \) on \( \Phi \). The table below gives the type of \( S \) for each of the special quadruples. (The following observation allows the type of \( S \) to be read off the Dynkin diagram for \( L \). If \( p = 2 \) and \( \mathcal{O} \cap \mathcal{O}' \neq \emptyset \) for distinct orbits \( \mathcal{O} \) and \( \mathcal{O}' \), then \( \mathcal{O} \subset \mathcal{O}' \) or vice versa. Without loss, assume the former. Then all but two of the summands in \( \gamma_{\mathcal{O}'} - \gamma_{\mathcal{O}} \) are orthogonal to all of the summands in \( \gamma_{\mathcal{O}} \), so that \( (\gamma_{\mathcal{O}}, \gamma_{\mathcal{O}'}) = (\gamma_{\mathcal{O}}, \gamma_{\mathcal{O}}) + (\gamma_{\mathcal{O}}, \gamma_{\mathcal{O}' - \mathcal{O}}) = 2 - 2 = 0. \)

Suppose \( L \) has type \( d_4 \) and \( p = 2 \). Let \( \theta \) be the standard graph automorphism of \( L \) of order 3 (it is inverted by \( \sigma \) under conjugation), and let \( S_\theta \) be the fixed points of \( \theta \) on \( L \). The \( \theta \)-orbits on \( \Phi \) consist of fixed points, which are also fixed by \( \sigma \), and the regular orbits, each of which is a union of a fixed point of \( \sigma \) and a regular \( \sigma \)-orbit. This implies that \( S_\theta \subseteq C \) and that \( S_\theta \cap N = 0 \). Thus there is a second possibility for \( S \) in this case, as indicated in Table 1, below.

The last statement is clear since \( M \) contains a copy of the Chevalley group associated to \( S \) and \( M \) becomes simple when the field is sufficiently large.

Notice that a further example of type \( a \) occurs here since \( a_3 \cong d_3 \).

**Table 1:** Special quadruples and covering algebras

\[
\begin{array}{cccccccc}
\text{L} & \text{M} & \text{dim} C & \text{dim} N & \text{dim} S & \text{dim} S & \text{dim} Z(S) & \text{dim} N \cap S \\
d_4 & G_2 & 14 & 7 & a_2 & 8 & 1 & 1 \\
e_6 & F_4 & 52 & 26 & d_4 & 28 & 2 & 2 \\
d_{n+1}, n \geq 3 & B_n & 2n^2 + n & 2n + 1 & d_n & n(2n - 1) & \{1, n \text{ odd} \} & 1 \\
a_{2n-1}, n \geq 2 & C_n & 2n^2 + n & 2n^2 - n - 1 & a_1 \oplus \ldots \oplus a_1 & 3n & n & n - 1 \\
d_4 & B_3 & 21 & 7 & g_2 & 14 & 0 & 0 \\
\end{array}
\]

**3.5 Corollary** For each of the following triples \((S, X, p)\), the Chevalley group of type \( X \) acts faithfully on the central quotient of the classical Lie algebra of type \( S \) in characteristic \( p \).

(i) \( S = a_3, X = G_2, p = 3 \).

(ii) \( S = d_4, X = F_4, p = 2 \).

(iii) \( S = d_n, X = B_n, p = 2 \).
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(iv) $S = g_2$, $X = B_3, p = 2$.

It is worth mentioning that the ideal $N \cap C$ of $C$ is associated to Chevalley basis elements for short roots in the root system inherited by $C$ from $\Phi$ and $\sigma$. This is immediate from our procedure since short roots are associated with sums over the regular orbits for $\sigma$. Similarly, it is immediate that $N \cap C$ is stable under the Chevalley group $R$ (and even $M$). This seems much easier than an argument within $C$ since a direct verification that the above subspace is an ideal and is stable under the Chevalley group, starting from the definitions of the Chevalley basis and the Chevalley generators $\varepsilon_r(t)$, would require a study of chains in the particular root systems and calculations of binomial coefficients.

We have shown that each of the algebras $S/Z(S)$ from the previous corollary admits, as automorphisms, a Chevalley group $M$ properly containing the Chevalley group associated to the classical Lie algebra $S$. We next proceed to determine the full automorphism group of $S/Z(S)$. We use algebraic group techniques to prove that the group is not bigger than the group $M$ when the field is algebraically closed. The situation is then fairly clear for general fields since we are dealing with Chevalley groups; the occurrence of outer diagonal automorphisms here depends on the particular field.

3.6 Lemma Suppose $\text{char}(K) = 2$ and that $L$ is a classical Lie algebra of type $d_n$, $n \geq 3$. Let $h_{ij}, h'_{ij}$ denote the elements $h_r$ of the Cartan subalgebra $h$ of $L$ given by roots $r = e_i + e_j, e_i - e_j$, in the usual notation [Bour]. If $n$ is odd, $Z(L)$ has basis $h_{ij} + h'_{ij}$, for any pair $i \neq j$. If $n$ is even, $Z(L)$ has basis consisting of an $h_{ij} + h'_{ij}$ as above, and $h_{12} + h_{34} + \ldots + h_{n-1,n}$. The usual graph automorphism of order 2 (associated to a determinant -1 diagonal transformation) commutates the second basis vector above to the first. As a module for the Weyl group ($\cong 2^{n-1} : \text{Sym}_n$), the maximal trivial quotient of the Cartan subalgebra is zero. In fact, the module structure for the Weyl group on $h$ has ascending factors of dimensions 1, $n - 1$ for $n$ odd and 1, 1, $n - 2$ for $n$ even.

Proof Exercise.

3.7 Lemma Suppose that $Z(S) \neq 0$ and that $S$ does not have type $a_1 \oplus \ldots \oplus a_1$ (so that $n \geq 3$ when $S$ has type $d_n$). Then $Z(S)$ is not complemented by an $R$-submodule.

Proof Suppose we are in the $a_2$ case. Let $u$ be a unipotent element of $R$ with a single Jordan block of size 3 on the natural module $V$. It has the same Jordan canonical form on the dual module, and on the tensor of these two representations, which contains the representation on $S$ as a constituent, it has three Jordan blocks of size 3. On $S/Z(S)$, of dimension 7, it has at least 3 dimensions of fixed points. If there were a complement invariant under $u$, $u$ would have fixed point subspace on $S$ of dimension at least 4, a contradiction.

All remaining cases satisfy $\text{char}(K) = 2$ and $S$ has type $d_n$, for some $n \geq 3$. The previous result deals with this case.

It is worth observing that the previous result gives a trivial proof of the $F_4$ and $G_2$ cases of the Hauptsatz of [Hiss], which determines the submodule structure of a classical Lie algebra with respect to its Chevalley group; for, if there were a submodule complementing the nontrivial ideal (of dimensions 26 and 7, respectively), we could descend to a covering subalgebra and its Chevalley group and contradict the previous lemma.
3.8 Theorem  If $K$ is algebraically closed then

(i) $\text{Aut}(a_2(K)/Z(a_2(K))) \cong G_2(K)$, for $\text{char}(K) = 3$.

(ii) $\text{Aut}(g_2(K)) \cong B_3(K)$, for $\text{char}(K) = 2$. Note that $Z(g_2(K))) = 0$ in this case.

(iii) $\text{Aut}(d_4(K)/Z(d_4(K))) \cong F_4(K)$, for $\text{char}(K) = 2$.

(iv) $\text{Aut}(d_n(K)/Z(d_n(K))) \cong B_n(K)$, for $\text{char}(K) = 2$, $n = 3$ or $n > 4$.

Proof  The approach to each case is similar, but the details vary. Let $(L, \sigma, M, K)$ be a special quadruple with associated covering subalgebra $S$; use the notations of (3.4). In each case, discussed below, we show that $M$ acts irreducibly on $Q/Z(Q)$, size $\Sigma$. This implies that a normal unipotent subgroup of $A$ is trivial, so any reductive subgroup centralizing $M$ (identified with its image in $A$) must be both scalar and a group of automorphisms, hence trivial. Therefore, $A$ is semisimple (and $M$ projects nontrivially to each factor).

We also note that a maximal torus $T$ of $M$ acts on $Q$ by pairwise distinct nontrivial linear characters at the root spaces and has fixed point subalgebra the Cartan subalgebra $h$. It follows that each of these spaces is left invariant by $C_A(T)$, as are the 1-dimensional subspaces of $h$ obtained by bracketing a root space and its negative. Since $C_A(T)$ effects a scalar on each invariant 1-space, indecomposability of the root system implies that $C_A(T)$ acts by a scalar-valued homomorphism on $h$. Also, if $E$ is any root space, the equation $[h, E] = E$ implies that $C_A(T)$ centralizes $h$. In particular, the Weyl group of $A$ acts on $h$.

An immediate corollary is that $A$ and $M$ have the same Lie rank; for otherwise, the kernel of the action of a torus of $C_A(T)$ on the above fundamental root spaces and their opposites would be positive dimensional. Finally, we deduce that $A$ is simple.

We settle all cases now with the observation that, in each case, there is no embedding of $M$ as a proper subgroup of a simple algebraic group with the same rank.

It remains to verify irreducibility of $M$ on $Q/Z(Q) \cong S/Z(S)$.

(i) $M \cong G_2(K)$ contains a subgroup of shape $2^3 \cdot SL(3, 2)$ [Gr 1990] which acts faithfully, hence irreducibly on the 7-dimensional module $Q$ (51.7)[CuRe].

(ii) Here, $M \cong B_3(K)$ acts on $Q \cong g_2(K)$. To show that this action is irreducible, it suffices to show that a $G_2(2)$-subgroup $X$ acts irreducibly. Let $x$ be a 3-central element of $X$ of order 3. Then $N_X((x)) \cong 3_{+}^{1+2} : [3 : 2]$ (the factor 8:2 is semidihedral), $x$ is real in $X$, and $x$ has fixed point subalgebra of dimension 8. If $P := O_3(N_X((x)))$, then any element of $P - (x)$ is real and has fixed point subalgebra of dimension 4; such elements form a single $N_X((x))$-conjugacy class. Orthogonality relations (done in characteristic 0 with Brauer characters) imply that $P$ has 0 fixed point subalgebra on $Q$. This also implies that $N_X((x))$ has irreducible constituents of degrees 8 and 6, and $x$ acts trivially on the 8-dimensional constituent. Since $(x^X)$ has index 2 in $X$, and since elements from the other class of elements of order 3 have fixed point subalgebras of dimension 4, it follows that this 8-dimensional constituent does not represent a composition factor for $X$. Therefore $X$ must act irreducibly on $Q$.

(iii) A subgroup of $M \cong F_4(K)$ of shape $3^3 : SL(3, 3)$ acts irreducibly on $Q$ due to the fact that its normal subgroup of order $3^3$ acts with 26 distinct nontrivial linear characters, which form an orbit under the action of $SL(3, 3)$. 

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(iv) If \( n = 3 \) then \( Q \cong g_2(K) \) by Lemma 3.2, and the result follows from case (ii). We therefore assume that \( n > 4 \). Let \( T \) be a maximal torus of \( M \cong B_n(K) \). Then, \( N_M(T) \) acts on the adjoint module \( C \) for \( M \) with irreducible constituents corresponding to the orbits of the Weyl group \( W = N_M(T)/T \) on the short and long roots, plus the constituents for the action of \( W \) on the Cartan subalgebra \( h \) of \( C \). When we pass to \( Q = C/N \), we factor out the span of the short root spaces and a 1-dimensional central ideal. When we pass to the full central quotient of \( Q \) we get an irreducible action of \( W \) on the image of \( h \) (see (3.7)). Since no Chevalley group element of \( M \), for \( r \) fixed by \( \sigma \), leaves invariant the image of \( h \) in \( Q \), we deduce irreducibility of \( M \) on \( Q/Z(Q) \), as required.

QED

Finally, we determine automorphism groups for most remaining central quotients of the covering subalgebras from Table 1 (we exclude just line 4). The essential case is that of an algebraically closed field.

3.9 Theorem Let \( S \) be a covering subalgebra from Table 1 with \( S/Z(S) \) simple, and let \( Z \) be a central ideal properly contained in \( Z(S) \). Assume that \( K \) is algebraically closed. Then \( \text{Aut}(S/Z)^0 \) is given by the natural action of \( R \), where \( R \) is the Chevalley group associated to \( S \), except in the cases

(i) \( S \) has type \( d_4 \) and \( Z \) is one of three particular one-dimensional ideals, in which case \( \text{Aut}(S/Z)^0 \) corresponds to one of the three natural type \( B_4 \) subgroups between the images of \( R \) and \( M \cong F_4(K) \) in \( \text{Aut}(S/Z(S))^0 \).

(ii) \( S \) has type \( d_n \), for even \( n \geq 2 \), \( \dim Z = 1 \) and \( \text{Aut}(S/Z) \cong B_n(K) \).

Proof Let \( Z \) be a central ideal proper in \( Z(S) \) and \( A = \text{Aut}(S/Z)^0 \). We deal with the various cases of \( Z(S) \neq 0 \) indicated in Table 1. Note that if \( Z_1 \) is any central ideal, the natural map from \( \{ \alpha \in \text{Aut}(S) \mid \alpha \text{ leaves } Z_1 \text{ invariant} \} \) to \( \text{Aut}(S/Z) \) is a monomorphism since \( S \) is perfect. Thus, identifying groups with their images in \( \text{Aut}(S/Z(S)) \), we get containments \( R \subseteq A \subseteq M \). We determine the middle group for all relevant \( Z \).

Case 1. \( S = a_2(K) \), for \( \text{char}(K) = 3 \).

Here, \( \dim Z = 0 \). Suppose that \( R < A \). Then, \( A = M \cong G_2(K) \). Let \( B \) be a subgroup of \( A \) isomorphic to \( 2^3 : SL(3, 2) \). Then, \( O_2(B) \) operates fixed point freely on \( S/Z(S) \) and so leaves invariant a unique complement \( V \) to \( Z(S) \) in \( S \). The subgroup \( A_V \) of \( A \) leaving \( V \) invariant is an algebraic subgroup of \( A \). If \( A_V \) is positive dimensional, then an argument similar to that of (3.8) shows that \( A_V \) must be simple since it contains \( B \) and has rank at most 2. Table 1 of (1.8)[Gr 1990] implies that \( A_V = M \), a contradiction to (3.7). We conclude that \( A_V \) is finite. The action of the 14-dimensional group \( A \) on a 6-dimensional space of complements in \( S \) to \( Z \) (this is equivalent to an action on a 6-dimensional affine space of vectors in the dual space of \( S \)) therefore has 0-dimensional fiber, a contradiction.

Case 2. \( S = d_n(K) \), for \( \text{char}(K) = 2 \).

Identify \( M \) with \( \text{Aut}(S/Z(S)) \). The image \( X \) of \( \text{Aut}(S/Z)^0 \) in \( M \) is an algebraic group between \( R \) and \( M \), so is either \( R \) or a natural \( B_n \) subgroup, or possibly \( M \cong F_4(K) \) when \( n = 4 \). The latter possibility for \( X \) is quickly eliminated by arguing as in Case 1, with a subgroup \( 3^3 : SL(3, 3) \) in the role of \( B \), using Table 2 of (1.8)[Gr 1990].
Consider the possibility that \( X \cong B_n(K) \). Then, \( X \) and \( R \) share a maximal torus and so the Weyl group of \( X \) acts on the image in \( S/Z \) of a Cartan subalgebra of \( S \). We assume that this action extends that of the Weyl group of \( R \) by a determinant -1 diagonal transformation in the usual description of the root system of type \( d_n \). When triality is present (for us, this means \( M \cong F_4(K) \)), we may need to use triality to assume the above.

If \( n \geq 2 \) is even, \( \dim Z = 0 \) is impossible, by the way the graph automorphism of \( d_n \) acts on the elements of (3.6), since we would then have a nontrivial homomorphism of \( B_n(K) \) to \( GL(2, K) \). So, if \( n \) is even, \( \dim Z = 1 \). This case occurs, by Table 1. What is needed now is the result that such a \( Z \) must be the span of \( h_{ij} + h'_{ij} \), in the notation of (3.6). This follows from the way the graph automorphism acts on \( h_{12} + \ldots + h_{n-1,n} \) modulo \( Z \).

If \( n \) is odd, \( n \geq 3 \) and the only possibility here is \( Z = 0 \). We show that \( X \cong B_n(K) \) is impossible. Let \( T \) be the 1-dimensional torus in \( X \) whose fixed point subgroup has semisimple part \( Y \) isomorphic to \( B_{n-1}(K) \). Without loss, \( Y \) corresponds to a subset of \( n-1 \) nodes of the Dynkin diagram for \( X \). We may assume that the subgroup generated by root groups associated to the long roots in the root system for \( Y \) is in \( R \). These roots therefore correspond to a subset of the given Chevalley basis and we have an associated classical subalgebra \( C_Q(T) \) of type \( d_{n-1} \). Since \( n-1 \) is even and since \( Y \) acts on \( C_Q(T) \), we have a contradiction to the previous paragraph, the case \( \dim Z = 0 \).

4. References


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