A vertex operator algebra related to E_8 with automorphism group $O^+(10,2)$

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Abstract. We study a particular VOA which is a subVOA of the E_8-lattice VOA and determine its automorphism group. Some of this group may be seen within the group $E_8(\mathbb{C})$, but not all of it. The automorphism group turns out to be the 3-transposition group $O^+(10,2)$ of order $2^{21}3^5 5^2 7.17.31$ and it contains the simple group $\Omega^+(10,2)$ with index 2. We use a recent theory of Miyamoto to get involutory automorphisms associated to conformal vectors. This VOA also embeds in the moonshine module and has stabilizer in $\mathbb{M}$, the monster, of the form $2^{10+16}.\Omega^+(10,2)$.

Hypotheses

We review some definitions, based on the usual definitions for the elements, products and inner products for lattice VOAs; see [FLM].

Notation 1.2. Φ is a root system whose components have types ADE, $\mathfrak{g}$ is a Lie algebra with root system Φ, $Q := Q_\Phi$, the root lattice and $V := V_Q := \mathbb{S}(\hat{H}_-) \otimes \mathbb{C}[Q]$ is the lattice VOA in the usual notation.

Remark 1.3. We display a few graded pieces of V ($\otimes$ is omitted, and here Q can be any even lattice). We write H_m for $H \otimes t^{-m}$ in the usual notation for lattice VOAs (2.1) and $Q_m := \{x \in Q \mid (x,x) = 2m\}$, the set of lattice vectors of type m.

$$V_0 = \mathbb{C}, \qquad V_1 = H_1,$$

$$V_2 = [S^2 H_1 + H_2] + H_1 \mathbb{C} Q_1 + \mathbb{C} Q_2,$$

$$V_3 = [S^3 H_1 + H_1 H_2 + H_3] + [S^2 H_1 + H_2]\mathbb{C} Q_1 + H_1 \mathbb{C} Q_2 + \mathbb{C} Q_3,$$

$$V_4 = [S^4 H_1 + S^2 H_1 H_2 + H_1 H_3 + S^2 H_2 + H_4] +$$

$$[S^3 H_1 + H_1 H_2 + H_3]\mathbb{C} Q_1 + [S^2 H_1 + H_2]\mathbb{C} Q_2 + H_1 \mathbb{C} Q_3 + \mathbb{C} Q_4.$$

Remark 1.4. Let F be a subgroup of $Aut(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra $V_1 = H_1 + \mathbb{C} Q_1$ with 0^{th} binary composition. The fixed points V^F of F on V form a subVOA. We have an action of $N(F)/F$ as automorphisms of this sub VOA.

Notation 1.5. *For the rest of this article, we take Q to be the E_8-lattice.* Take F to be a $2B$-pure elementary abelian 2-group of rank 5 in $Aut(\mathfrak{g}) \cong E_8(\mathbb{C})$; it is fixed point free. Let $E := F \cap T$ where T is the standard torus and where F is chosen to make $rank(E) = 4$. Let $\theta \in F \setminus E$; we arrange for θ to interchange the standard Chevalley generators x_α and $x_{-\alpha}$. See [Gr91]. The Chevalley generator x_α corresponds to the standard generator e^α of the lattice VOA V_Q.

Notation 1.7. $L := Q^{[E]} \cong \sqrt{2}Q$ denotes the common kernel of the lattice characters associated to the elements of E; in the [Carter] notation, these characters are $h^{-1}(E)$; in the root lattice modulo 2, they correspond to the sixteen vectors in a maximal totally singular subspace. Then

$$V_1^F = 0 \tag{1.7.1}$$

and

$$V_2^F = S^2H_1 + 0 + \mathbb{C}L_2^\theta, \tag{1.7.2}$$

where the latter summand stands for the span of all $e^\lambda + e^{-\lambda}$, where λ runs over all the $15 \cdot 16 = 240$ norm 4 lattice vectors in L. Thus, V_2^F has dimension $\binom{9}{2} + \frac{240}{2} = 36 + 120 = 156$ and has a commutative algebra structure invariant under $N(F) \cong 2^{5+10} \cdot GL(5,2)$. We note that $N(F)/F \cong 2^{10}{:}GL(5,2)$ [Gr76][CoGr][Gr91].

We will show (6.10) that $Aut(V^F) \cong O^+(10,2)$.

2. Inner Product.

Definition 2.1. The inner product on S^nH_m is $\langle x^n, x^n\rangle = n!m^n\langle x,x\rangle^n$. This is based on the adjointness requirement for $h \otimes t^k$ and $h \otimes t^{-k}$ (see (1.8.15), FLM,p.29). When $k > 0$, $h \otimes t^{-k}$ acts like multiplication by $h \otimes t^{-k}$ and, when h is a root, $h \otimes t^k$ acts like k times differentiation with respect to h.

When $n = 2$, this means $\langle x^2, x^2\rangle = 2m^2\langle x,x\rangle$. In V_2^F, $m = 1$.

Definition 2.2. *The Symmetric Bilinear Form.* Source: [FLM], p.217. This form is associative with respect to the product (Section 3). We write H for H_1. The set of all g^2 and x_α^+ spans V_2.

$$\langle g^2, h^2\rangle = 2\langle g,h\rangle^2, \tag{2.2.1}$$

whence

$$\langle pq, rs\rangle = \langle p,r\rangle\langle q,s\rangle + \langle p,s\rangle\langle q,r\rangle, \text{ for } p,q,r,s \in H. \tag{2.2.2}$$

$$\langle x_\alpha^+, x_\beta^+\rangle = \begin{cases} 2 & \alpha = \pm\beta \\ 0 & \text{else} \end{cases} \tag{2.2.3}$$

$$\langle g^2, x_\beta^+\rangle = 0. \tag{2.2.4}$$

Notation 2.3. In addition, we have the distinguished Virasoro element ω and identity $\mathbb{I} := \frac{1}{2}\omega$ on V_2 (see Section 3). If h_i is a basis for H and h_i^* the dual basis, then $\omega = \frac{1}{2}\sum_i h_i h_i^*$.

Remark 2.4.

$$\langle g^2, \omega\rangle = \langle g, g\rangle \tag{2.4.1}$$

$$\langle g^2, \mathbb{I}\rangle = \frac{1}{2}\langle g, g\rangle \tag{2.4.2}$$

$$\langle \mathbb{I}, \mathbb{I}\rangle = dim(H)/8 \tag{2.4.3}$$

$$\langle \omega, \omega\rangle = dim(H)/2 \tag{2.4.4}$$

If $\{x_i \mid i = 1, \ldots \ell\}$ is an ON basis,

$$\mathbb{I} = \frac{1}{4}\sum_{i=0}^{\ell} x_i^2 \tag{2.4.5}$$

$$\omega = \frac{1}{2}\sum_{i=0}^{\ell} x_i^2. \tag{2.4.6}$$

3. The Product on V_2^F.

Definition 3.1. *The product on* V_2^F comes from the vertex operations. We give it on standard basis vectors, namely $xy \in S^2H_1$, for $x, y \in H_1$ and $v_\lambda := e^\lambda + e^{-\lambda}$, for $\lambda \in L_2$. Note that (3.1.1) give the Jordan algebra structure on S^2H_1, identified with the space of symmetric 8×8 matrices, and with $\langle x, y\rangle = \frac{1}{8}tr(xy)$. The function ε below is a standard part of notation for lattice VOAs.

$$x^2 \times y^2 = 4\langle x, y\rangle xy, \qquad pq \times y^2 = 2\langle p, y\rangle qy + 2\langle q, y\rangle py, \tag{3.1.1}$$
$$pq \times rs = \langle p, r\rangle qs + \langle p, s\rangle qr + \langle q, r\rangle ps + \langle q, s\rangle pr;$$

$$x^2 \times v_\lambda = \langle x, \lambda\rangle^2 v_\lambda, \qquad xy \times v_\lambda = \langle x, \lambda\rangle\langle y, \lambda\rangle v_\lambda \tag{3.1.2}$$

$$v_\lambda \times v_\mu = \begin{cases} 0 & \langle\lambda, \mu\rangle \in \{0, \pm 1, \pm 3\}; \\ \varepsilon\langle\lambda, \mu\rangle v_{\lambda+\mu} & \langle\lambda, \mu\rangle = -2; \\ \lambda^2 & \lambda = \mu. \end{cases} \tag{3.1.3}$$

Convention 3.2. Recall that $L = Q^{[E]}$. Since $(L, L) \leq 2\mathbb{Z}$, we may and do assume that ε is trivial on $L \times L$.

4. Some Calculations with Linear Combinations of the v_λ.

Notation 4.1. For a subset M of H, there is a unique element ω_M of S^2H which satisfies (1) $\omega_M \in S^2(span(M))$; (2) for all $x, y \in span(M)$, $\langle x, y\rangle = \langle \omega_M, xy\rangle$. We define $\mathbb{I}_M := \frac{1}{2}\omega_M$. If M and N are orthogonal sets, we have $\omega_{M\cup N} = \omega_M + \omega_N$. Define $\omega'_M := \omega - \omega_M$ and $\mathbb{I}'_M := \mathbb{I} - \mathbb{I}_M$. This element can be written as $\omega_M = \frac{1}{2}\sum_i x_i^2$, where the x_i form an orthonormal basis of $span(M)$. We have $\langle \omega_M, \omega_M\rangle = \frac{1}{2} dim\ span(M)$ and $\langle \mathbb{I}_M, \mathbb{I}_M\rangle = \frac{1}{8} dim\ span(M)$. Also, $\langle \omega_M, xy\rangle = \langle \omega_M, x'y'\rangle = \langle \omega, x'y'\rangle$, where priming denotes orthogonal projection to $span(M)$.

Notation 4.2. $e^\pm_\lambda := f^\mp_\lambda := \frac{1}{32}[\lambda^2 \pm 4v_\lambda]$, $e_\lambda = e^+_\lambda$, $f_\lambda := e^-_\lambda$. If $a \in \mathbb{Z}$ or $\mathbb{Z}_2$, define $e_{\lambda,a}$ to be e^+_λ or e^-_λ, as $a \equiv 0, 1 (mod\ 2)$, respectively; see (4.7). Also, let $e'_{\lambda,a} = e_{\lambda,a+1}$. We define $e_{\lambda,\mu}$ to be $e_{\lambda,a}$, where a is $\frac{1}{2}\langle\lambda, \mu\rangle$ in case μ is a vector in L, and a is $[\hat{\lambda}, \mu]$, where $[.,.]$ is the nonsingular bilinear form on $Hom(L, \{\pm 1\})$ gotten from $2\langle .,.\rangle$ by thinking of $Hom(L, \{\pm 1\})$ as $\frac{1}{2}L/L$ and where $\hat{\lambda}$ is the character gotten by reducing the inner product with $\frac{1}{2}\lambda$ modulo 2. Finally. let q be the quadratic form on $Hom(L, \{\pm 1\})$ gotten by reducing $x \mapsto \langle x, x\rangle$ modulo 2, for $x \in \frac{1}{2}L$.

Lemma 4.3. *(i) The $e^\pm_\lambda$ are idempotents.*

(ii) $\langle e^\pm_\lambda, e^\pm_\mu\rangle = \begin{cases} \frac{1}{16} & \lambda = \mu; \\ \frac{1}{128} & \langle\lambda,\mu\rangle = -2; \\ 0 & \langle\lambda,\mu\rangle = 0. \end{cases}$

(iii) $\langle e^\pm_\lambda, e^\mp_\mu\rangle = \begin{cases} 0 & \lambda = \mu; \\ \frac{1}{128} & \langle\lambda,\mu\rangle = -2; \\ 0 & \langle\lambda,\mu\rangle = 0. \end{cases}$

Proof. (i) $(e^\pm_\lambda)^2 = \frac{1}{1024}[4\cdot 4\lambda^2 + 16\lambda^2 \pm 8\cdot 4^2 v_\lambda] = e^\pm_\lambda$. (ii) and (iii) follow trivially from (2.2).

Notation 4.4. For finite $X \subseteq L$, define $s(X) := \sum_{x\in\pm X/\{\pm 1\}} x^2$.

Lemma 4.5. *If* $X \subseteq L_2$, $\langle \omega, s(X)\rangle = 4|(\pm X)/\{\pm 1\}|$ *and so* $s(L_2) = \frac{|L_2|}{2}\omega = 120\omega$.

Proof. (2.2.5)

Corollary 4.6. *(i) For* $\alpha \in L_2$, $\langle \omega_\alpha, s(\alpha)\rangle = \langle \omega, s(\alpha)\rangle = 4$ *and* $\langle \omega_\alpha, \omega_\alpha\rangle = \frac{1}{2}$, *whence* $s(\alpha) = 8\omega_\alpha = 16\mathbb{I}_\alpha$ *and* $\mathbb{I}_\alpha = \frac{1}{16}\alpha^2$.

(ii) $\langle \omega_{E_7}, s(\Phi_{E_7})\rangle = \langle \omega, s(\Phi_{E_7})\rangle = 63$, *whence* $s(\Phi_{E_7}) = 18\omega_{\Phi_{E_7}}$;

(iii) $\langle \omega, s(\Phi_{D_8})\rangle = 56$, *whence* $s(\Phi_{D_8}) = 56\omega_{\Phi_{E_7}}$.

Notation 4.7. *For $\varphi \in Hom(L, \{\pm 1\})$, define $f(\varphi) := \sum_{\lambda \in L_2/\{\pm 1\}} \varphi(\lambda) v_\lambda$, $u(\varphi) := \sum_{\lambda \in L_2/\{\pm 1\}} \varphi(\lambda)\lambda^2$ and $e(\varphi) := \frac{1}{16}\mathbb{I} + \frac{1}{64} f(\varphi)$. These arguments may come from other domains, as in (4.2), and we allow mixing as in $e(\varphi\lambda)$, for a character φ and lattice vector λ. We prove later that $e(\varphi)$ is an idempotent.*

Lemma 4.8. *Let $r, s \in L$, $a, b \in \mathbb{Z}$ and let*

$$n(r,s,a,b) := \frac{1}{2}\ |\{t \in \Phi | \langle r,t\rangle \equiv 2a (mod\ 2), \langle s,t\rangle \equiv 2b (mod\ 2)\}|.$$

(i) Suppose that the images of r and s in $L/2L$ are nonzero and distinct. The values of $n(r,s,a,b)$ depend only on the isometry type of the images of the ordered pair $\langle r,s\rangle$ in $L/2L$ and are listed below:

$\frac{1}{4}\langle r,r\rangle$	$\frac{1}{4}\langle s,s\rangle$	$\frac{1}{2}\langle r,s\rangle$	$2n(rs00)$	$2n(rs01)$	$2n(rs10)$	$2n(rs11)$
0	0	0	48	64	64	64
0	0	1	56	56	56	72
0	1	0	64	48	64	64
0	1	1	56	56	72	56
1	0	0	64	64	48	64
1	0	1	56	72	56	56
1	1	0	64	64	64	48
1	1	1	72	56	56	56.

(ii) If $s = 0$ and $\langle r,r\rangle = 4$, then $2n(r,s,0,0) = 128$ and $2n(r,s,1,0) = 112$. If $s = 0$ and $\langle r,r\rangle = 8$, then $2n(r,s,0,0) = 112$ and $2n(r,s,1,0) = 128$.

Lemma 4.9. *The $f(\varphi)$, as φ ranges over all nonsingular characters of L of order 2, form a basis for $\mathbb{C}L_2^\theta$.*

Proof. Use the action of the subgroup of the Weyl group stabilizing the maximal totally singular subspace $L/2Q$ of $Q/2Q$ (its shape is $2^{-5}2^8.2_+^{1+6}.GL(4,2)$); it also stabilizes the maximal totally singular subspace $2Q/2L$ of $L/2L$ (halve the quadratic form on L, then reduce modulo 2). Since W_{E_8} induces the group $O^+(8,2)$ on $Q/2Q$, Witt's theorem implies that the stabilizer of a maximal isotropic subspace is transitive on the nonsingular vectors outside it. The action of this group on $L/2L$ has the analogous property.

Notation 4.10. $u(\varphi) := \sum_{\lambda \in L_2/\{\pm 1\}} \varphi(\lambda)\lambda^2$. *This also makes sense for $\varphi \in L$ by the identification in (4.2).*

Proposition 4.11. *Let $\alpha \in Hom(L, \{\pm 1\})$.*

(i)

$$\langle u(\alpha), \omega\rangle = \sum \alpha(\lambda)\lambda^2 = \begin{cases} 480 & \alpha = 1; \\ -32 & \alpha singular; \\ 32 & \alpha nonsingular. \end{cases}$$

(ii)

$$u(\alpha) = \begin{cases} 240\mathbb{I} & \textit{if } \alpha\textit{=1;} \\ -16\mathbb{I} & \textit{if } \alpha \textit{ is singular;} \\ -208\mathbb{I}_\alpha + 48\mathbb{I}'_\alpha = -256\mathbb{I}_\alpha + 48\mathbb{I} = -16\alpha^2 + 48\mathbb{I} & \textit{if } \alpha \textit{ is nonsingular;} \end{cases}$$

(in the third case, α is taken to be a norm 4 lattice vector in L^θ; it is well defined up to its negative, and this suffices). Their respective norms are 57600, 256 and $\frac{1}{8}208^2 + \frac{7}{8}48^2 = 7424 = 2^8 29$.

Proof. We deal with cases, making use of inner product results (2.2) and (2.3); at once, we get (i). If $u(\alpha)$ were known to be a multiple of $\mathbb{I}$, this inner product information would be enough to determine $u(\alpha)$. This is so for $u(1)$ since the linear group (isomorphic to the Weyl group of E_8) stabilizing L_2 is irreducible and so fixes a subspace of dimension just 1 in the symmetric square of H. It follows that $u(1) = 240\mathbb{I}$.

Notice that in all cases $u(\alpha) = 2u'(\alpha) - u(1)$, where $u'(\alpha) := \sum_{\lambda \in \Phi'/\{\pm 1\}} \lambda^2$ and $\Phi' := \{\lambda \in \Phi \mid \alpha(\lambda) = 1\}$.

Now to evaluate $u' := u'(\alpha)$ for $\alpha \neq 1$. If Φ' has type D_8, we have an irreducible group as above and conclude that $u'(\alpha) = b\mathbb{I}$, where $b = \langle u', \mathbb{I}\rangle = \frac{1}{2}\langle u', \omega\rangle = \frac{1}{2}56 \cdot 4 = 112$. If Φ' has type A_1E_7, we have a reducible group with two constituents and conclude that $u' = c\mathbb{I}_\alpha + d\mathbb{I}_{\alpha^\perp}$, where we interpret α as an element of L_2 and moreover as a root in the A_1-component of Φ'. Since $\langle \mathbb{I}_\alpha, \mathbb{I}_\alpha\rangle = \frac{1}{8}$ and $\langle \alpha^2, \alpha^2\rangle = 32$, $c = 16$. Since $\mathbb{I} = \mathbb{I}_\alpha + \mathbb{I}_{\alpha^\perp}$, $\frac{1}{8}c + \frac{7}{8}d = \langle u', \mathbb{I}\rangle = 128$, whence $d = 144$. Thus, $2u' - 240\mathbb{I} = 32\mathbb{I}_\alpha + 288\mathbb{I}'_\alpha - 240\mathbb{I} = -208\mathbb{I}_\alpha + 48\mathbb{I}'_\alpha = -256\mathbb{I}_\alpha + 48\mathbb{I}$.

Lemma 4.12. $f(\varphi) \times f(\psi) =$

$$\begin{cases} (-1)^{1+q(\varphi\psi)}4(f(\varphi) + f(\psi)) + (-1)^{1+\langle\varphi,\psi\rangle}64v_{\varphi\psi} + u(\varphi\psi) \\ \qquad \text{if } \varphi \neq \psi\text{; furthermore, this equals} \\ \qquad -4(f(\varphi) + f(\psi)) - 16\mathbb{I} \\ \qquad\qquad \text{if } \varphi\psi \text{ singular; and equals} \\ \qquad 4(f(\varphi) + f(\psi)) + 48\mathbb{I} - 512e_{\alpha,\langle\varphi,\psi\rangle} \\ \qquad\qquad \text{if } \varphi\psi \text{ nonsingular;} \\ 56f(\varphi) + u(1) \\ \qquad\qquad \text{if } \varphi = \psi. \end{cases}$$

Proof. The left side is

$$\sum_\lambda \sum_{\mu:\langle\mu,\lambda\rangle=-2} \varphi(\lambda)\psi(\mu)v_{\lambda+\mu} + u(\varphi\psi) = (\text{for } \nu = \lambda + \mu) \sum_\nu \psi(\nu) \sum_{\lambda:\langle\nu,\lambda\rangle=2} (\varphi\psi)(\lambda)v_\nu$$

$$= \sum_\nu \psi(\nu)(n(\nu, \varphi\psi, 1, 0) - n(\nu, \varphi\psi, 1, 1))v_\nu + u(\varphi\psi).$$

We use (4.8) and (4.9). The coefficent of v_ν is 0 if $\varphi\psi(\nu) \equiv 1(mod\ 2)$. If $\varphi\psi(\nu) \equiv 0(mod\ 2)$, then $\varphi(\nu) = \psi(\nu)$; the coefficient is $56\psi(\nu)$ if $\varphi\psi = 1$, $(-1)^{1+q(\varphi\psi)}8$ if $\varphi\psi \neq 1$ or ν and, if $\varphi\psi = \nu$, it is $-56(-1)^{\langle\varphi,\psi\rangle}$.

Corollary 4.13. $e(\varphi)^2 = e(\varphi)$.

The 256 $f(\varphi)$ *live in* $\mathbb{C}L^\theta$, *a space of dimension 120, so they are linearly dependent. There is a natural subset which forms a basis.*

Proposition 4.14. *If* $\varphi\psi$ *is singular,* $f(\varphi) \times f(\psi) = -4(f(\varphi)+f(\psi)) - 16\mathbb{I}$ *and* $e(\varphi) \times e(\psi) = 0$.

Proof. It suffices to show that $(4\mathbb{I}+f(\varphi)) \times (4\mathbb{I}+f(\psi)) = 0$, or $16\mathbb{I} + 4(f(\varphi) + f(\psi)) + 4(-1)^{1+q(\varphi\psi)}(f(\varphi)+f(\psi)) + u(\varphi\psi) = 0$. This follows from $q(\varphi\psi) = 0$ and $u(\varphi\psi) = -16\mathbb{I}$; see (4.12.i).

Lemma 4.15. *(i)* $\langle f(\varphi), f(\psi)\rangle = \begin{cases} 240 & \text{if } \varphi = \psi; \\ -16 & \text{if } \varphi\psi \text{ singular;} \\ 16 & \text{if } \varphi\psi \text{ nonsingular.} \end{cases}$

(ii) $\langle e(\varphi), e(\psi)\rangle = \begin{cases} \frac{1}{16} & \text{if } \varphi = \psi; \\ 0 & \text{if } \varphi\psi \text{ singular;} \\ \frac{1}{128} & \text{if } \varphi\psi \text{ nonsingular.} \end{cases}$

Proof. (i) This inner product is $2\sum_\lambda \varphi\psi(\lambda)$, so consider the cases that $\varphi\psi$ is 1, singular or nonsingular. One can also use (4.12) and associativity of the form. We leave (ii) as an exercise, with (i) and (2.2).

Theorem 4.16. *The* $2e(\varphi)$ *are conformal vectors of conformal weight (=central charge)* $\frac{1}{2}$.

Proof. By (4.10) and [Miy], Theorem 4.1, these are conformal vectors. Fix φ. Choose a maximal, totally singular subspace, J, of L modulo $2L$. Let $\mathfrak{J}$ be the set of distinct linear characters of L which contain J in their kernel. The $e(\psi)$, for $\psi \in \varphi\mathfrak{J}$, are pairwise orthogonal idempotents (4.12) which sum to $\mathbb{I}$ (to prove this, use the orthogonality relations for this set of 16 distinct characters). We use the fact that conformal weight of $2e(\varphi)$ is at least $\frac{1}{2}$ (see Proposition 6.1 of [Miy]). Since their conformal weights add to 8, the conformal weight of ω, we are done.

Notation 4.17. In an integral lattice, an element of norm 2 is called a $root$ and an element of norm 4 is called a $quoot$ (suggested by the term "quartic" for degree 4).

Notation 4.18. The idempotents $e^{\pm}_\lambda$ are called $idempotents\ of\ quoot\ type$ or $quooty\ idempotents$ and the $e(\varphi)$ are called idempotents of $tout\ type$ or $tooty\ idempotents$ (suggested by "tout" or "tutti"). The set of all such is denoted $\mathfrak{QI}$, $\mathfrak{TI}$, respectively. Set $\mathfrak{QTI} := \mathfrak{QI} \cup \mathfrak{TI}$.

5. Eigenspaces.

Notation 5.1. For an element x of a ring, ad_x, $ad(x)$ denotes the endomorphism: right multiplication by x. If the ring is a finite dimensional algebra over a field, the $spectrum\ of\ x$ means the spectrum of the endomorphism $ad(x)$.

The main result of this section is the following.

Theorem 5.2. *If* e *is one of the idempotents* e_λ, f_λ *or* $e(\varphi)$ *of Section 4, its spectrum is* $(1^1, \frac{1}{4}^{35}, 0^{120})$.

We prove (5.2) in steps, treating the quooty and tooty cases separately.

Table 5.3. The action of $ad(e_\lambda)$ on a spanning set. Recall that $e_\lambda = \frac{1}{32}[\lambda^2 + 4v_\lambda]$.

vector	image under $ad(e_\lambda)$	dim.
μ^2	$\frac{1}{32}[4\langle\lambda,\mu\rangle\lambda\mu + 4\langle\lambda,\mu\rangle^2 v_\lambda] = \frac{1}{8}[\langle\lambda,\mu\rangle\lambda\mu + \langle\lambda,\mu\rangle^2 v_\lambda]$	36
$\mu\nu$	$\frac{1}{32}[2\langle\lambda,\mu\rangle\lambda\nu + 2\langle\lambda,\nu\rangle\lambda\mu + 4\langle\lambda,\mu\rangle\langle\lambda,\nu\rangle v_\lambda] = \frac{1}{16}[\langle\lambda,\mu\rangle\lambda\nu + \langle\lambda,\nu\rangle\lambda\mu + 2\langle\lambda,\mu\rangle\langle\lambda,\nu\rangle v_\lambda]$	36
λ^2	$\frac{1}{32}[16\lambda^2 + 64v_\lambda] = 16e_\lambda$	1
$\lambda h,\ \langle\lambda,h\rangle = 0$	$\frac{1}{32}8\lambda h = \frac{1}{4}\lambda h$	7
$gh,\ \langle g,\lambda\rangle = \langle h,\lambda\rangle = 0$	0	28
v_λ	$4\frac{1}{32}[4\lambda^2 + 16v_\lambda] = 4e_\lambda$	1
$v_\mu,\ \langle\lambda,\mu\rangle = 0$	0	63
$v_\mu,\ \langle\lambda,\mu\rangle = -2$	$\frac{1}{32}[4v_{\lambda+\mu} + 4v_\mu] = \frac{1}{8}[v_{\lambda+\mu} + v_\mu]$	56

Table 5.4. The eigenspaces of $ad(e_\lambda)$.

eigenvalue	basis element(s)	dimension
1	e_λ	1
$\frac{1}{4}$	$\lambda h,\ \langle\lambda,h\rangle = 0$	7
$\frac{1}{4}$	$v_{\lambda+\mu} + v_\mu,\ \langle\lambda,\mu\rangle = -2$	28
0	$v_{\lambda+\mu} - v_\mu,\ \langle\lambda,\mu\rangle = -2$	28
0	$gh,\ \langle g,\lambda\rangle = \langle h,\lambda\rangle = 0$	28
0	$v_\mu,\ \langle\lambda,\mu\rangle = 0$	63
0	f_λ	1

Table 5.5. The action of $ad(f_\lambda)$ on a spanning set. Recall that $f_\lambda = \frac{1}{32}[\lambda^2 - 4v_\lambda] = -e_\lambda + \frac{1}{16}\lambda^2$, so the table below may be deduced from Table (5.3) and (3.1.1).

vector	image under $ad(f_\lambda)$	dimension
μ^2	$\frac{1}{4}\langle\lambda,\mu\rangle\lambda\mu - \frac{1}{8}[\langle\lambda,\mu\rangle\lambda\mu + \langle\lambda,\mu\rangle^2 v_\lambda] = \frac{1}{8}[\langle\lambda,\mu\rangle\lambda\mu - \langle\lambda,\mu\rangle^2 v_\lambda]$	36
$\mu\nu$	$\frac{1}{8}[\langle\lambda,\mu\rangle\lambda\nu + \langle\lambda,\nu\rangle\lambda\mu] - \frac{1}{16}[\langle\lambda,\mu\rangle\lambda\nu + \langle\lambda,\nu\rangle\lambda\mu + 2\langle\lambda,\mu\rangle\langle\lambda,\nu\rangle v_\lambda] = \frac{1}{16}[\langle\lambda,\mu\rangle\lambda\nu + \langle\lambda,\nu\rangle\lambda\mu - 2\langle\lambda,\mu\rangle\langle\lambda,\nu\rangle v_\lambda]$	36
λ^2	$\lambda^2 - 16e_\lambda = 16f_\lambda$	1
$\lambda h,\ \langle\lambda,h\rangle = 0$	$\frac{1}{32}8\lambda h = \frac{1}{4}\lambda h$	7
$gh,\ \langle g,\lambda\rangle = \langle h,\lambda\rangle = 0$	0	28
v_λ	$v_\lambda - 4e_\lambda = -4f_\lambda$	1
$v_\mu,\ \langle\lambda,\mu\rangle = 0, \pm 1$	0	63
$v_\mu,\ \langle\lambda,\mu\rangle = -2$	$\frac{1}{4}v_\mu - \frac{1}{8}[v_{\lambda+\mu} + v_\mu] = \frac{1}{8}[-v_{\lambda+\mu} + v_\mu]$	56

Table 5.6. The eigenspaces of $ad(f_\lambda)$.

eigenvalue	basis element(s)	dimension
1	f_λ	1
$\frac{1}{4}$	$\lambda h,\ \langle\lambda,h\rangle = 0$	7
0	$v_{\lambda+\mu} + v_\mu,\ \langle\lambda,\mu\rangle = -2$	28
$\frac{1}{4}$	$v_{\lambda+\mu} - v_\mu,\ \langle\lambda,\mu\rangle = -2$	28
0	$gh,\ \langle g,\lambda\rangle = \langle h,\lambda\rangle = 0$	28
0	$v_\mu,\ \langle\lambda,\mu\rangle = 0$	63
0	e_λ	1

Table 5.7. The action of $ad(f(\varphi))$ on a spanning set.

vector	image under $ad(f(\varphi))$	dim.
$f(\varphi)$	$56f(\varphi)+u(1)$	1
$f(\psi), \psi\varphi$ singular	$-4(f(\varphi)+f(\psi))-16\mathbb{I}$	120
$f(\psi), \psi\varphi$ nonsingular	$4(f(\varphi)+f(\psi))+48\mathbb{I}-512e_{\alpha,\langle\varphi,\psi\rangle}$	120
$u(\alpha), \alpha$ nonsingular	$\sum_\mu \varphi(\mu)\sum_\lambda \alpha(\lambda)\langle\lambda,\mu\rangle^2 v_\mu =$ $\sum_\mu \varphi(\mu)[48-16\langle\mu,\alpha\rangle^2]v_\mu$	36
$\mathbb{I}$	$f(\varphi)$	1

Proofs (5.7.1). Proofs of the above are straightforward. We give a proof only of the formula for $\xi := f(\varphi)\times u(\alpha)$. Clearly, ξ is a linear combination of the v_λ, so we just get its coefficent at v_λ as $\frac{1}{2}\langle\xi, v_\lambda\rangle$. By associativity of the form, this is $\frac{1}{2}\langle u(\alpha), f(\varphi)\times\ v_\lambda\rangle = \frac{1}{2}\langle u(\alpha), \varphi(\lambda)\lambda^2\rangle$. By (4.12.ii), we have an expression for $u(\alpha)$. Since $\langle\mathbb{I},\lambda^2\rangle = 2$ and $\langle\mathbb{I}_\alpha,\lambda^2\rangle = \frac{1}{2}\langle\lambda,\alpha\rangle^2\langle\alpha,\alpha\rangle^{-1} = \frac{1}{8}\langle\lambda,\alpha\rangle^2$, the respective cases of (4.12.ii) lead to $\frac{1}{2}\langle u(\alpha),\varphi(\lambda)\lambda^2\rangle = \varphi(\lambda)240, -\varphi(\lambda)16$ and $\varphi(\lambda)[48-16\langle\lambda,\alpha\rangle^2]$. Only the latter case is recorded in the table since $u(\alpha)$ is otherwise a multiple of $\mathbb{I}$.

Table 5.8. The action of $ad(e(\varphi))$ on a spanning set. Recall that $e(\varphi) = \frac{1}{16}\mathbb{I} + \frac{1}{64}f(\varphi)$. We use the notation $\alpha := \varphi\psi$, when $\varphi\psi$ is nonsingular. Note that the set of such α^2 span $S^2(H)$.

vector	image under $ad(e(\varphi))$	dim.
$f(\varphi)$	$\frac{15}{16}f(\varphi) + \frac{15}{4}\mathbb{I}$	1
$f(\psi), \varphi\psi$ singular	$-\frac{1}{16}f(\psi) - \frac{1}{4}\mathbb{I}$	120
$f(\psi), \varphi\psi$ nonsingular	$4e(\varphi) + 8e(\psi) - 8e_{\alpha,\langle\varphi,\psi\rangle}$	120
α^2 if $\alpha := \varphi\psi$ nonsingular	$2[e(\varphi) - e(\psi)] + 2e_{\alpha,\varphi(\alpha)}$	36
$u(\alpha), \alpha$ nonsingular	$\frac{1}{16}u(\alpha) + \sum_\mu \varphi(\mu)[\frac{3}{4} - \frac{1}{4}\langle\mu,\alpha\rangle^2]v_\mu$	36
$\mathbb{I}$	$e(\varphi)$	1
$e(\varphi)$	$e(\varphi)$	1
$e(\psi)$ if $\varphi\psi$ singular	0	120
$e(\psi)$ if $\alpha := \varphi\psi$ nonsingular	$2^{-3}[e(\varphi) + e(\psi) - e_{\alpha,\langle\varphi,\psi\rangle}]$	120
$v_\alpha = 4(e^+_\alpha - e^-_\alpha)$	$\varphi(\alpha)\frac{1}{2}[e_{\alpha,\langle\varphi,\psi\rangle} + e(\varphi) - e(\psi)]$	120
$e_{\alpha,\varphi}$	$\frac{1}{8}[e_{\alpha,\langle\varphi,\psi\rangle} + e(\varphi) - e(\psi)]$	35
$e'_{\alpha,\varphi}$	0	120

Table 5.9. Eigenspaces of $ad(e(\varphi))$. In the table, we use the convention that $\alpha := \varphi\psi$ is nonsingular. Recall that $e^\pm_\lambda = \frac{1}{32}(\lambda^2 \pm 4e_\lambda)$. Recall that $v_\alpha = 4(e^+_\alpha + e^-_\alpha)$.

eigenvalue	basis elements	dimension
1	$e(\varphi)$	1
0	$e'_{\alpha,\varphi}$	120
$\frac{1}{4}$	$-e_{\alpha,\varphi} + e(\psi)$	35

Table 5.10. Action of Idempotents on Idempotents. Recall the definitions $e^{\pm}_{\lambda} = \frac{1}{32}[\lambda^2 \pm 4v_\lambda]$, $e_{\lambda,\varphi} = e_{\lambda,\langle\varphi,\lambda\rangle}$, $e(\varphi) = \frac{1}{16}\mathbb{I} + \frac{1}{64}f(\varphi)$. In expressions below, a and b are integers modulo 2.

$$e_{\lambda,a} \times e_{\mu,b} = \begin{cases} 0 & \text{if } \langle\lambda,\mu\rangle = 0 \\ \quad 2^{-10}[-8\lambda\mu + 16((-1)^a v_\lambda + & \\ \quad (-1)^b v_\mu + 16(-1)^{a+b} v_{\lambda+\mu}] = & \\ \quad 2^{-10}[-4(\lambda+\mu)^2 - (-1)^{a+b} 4v_{\lambda+\mu} & \\ \quad +4((\lambda^2 + 4(-1)^a v_\lambda) + & \\ \quad 4(\mu^2 + 4(-1)^b v_\mu)] = & \\ 2^{-3}[e_{\lambda+\mu,a+b+1} + e_{\lambda,a} + e_{\mu,b}] & \text{if } \langle\lambda,\mu\rangle = -2 \\ e_{\lambda,a} & \text{if } (\langle\lambda,\mu\rangle, (-1)^{a+b}) \\ & \text{= (4,0), (-4,1)} \\ 0 & \text{if } (\langle\lambda,\mu\rangle, (-1)^{ab}) = \\ & \text{(4,1), (-4,0)} \end{cases}$$

$$e(\varphi) \times e(\psi) = \begin{cases} e(\varphi) & \text{if } \varphi = \psi \\ 0 & \text{if } \varphi\psi \text{ singular} \\ 2^{-3}[e(\varphi) + e(\psi) - e_{\varphi\psi,\varphi}] & \text{if } \varphi\psi \text{ nonsingular} \end{cases}$$

$$e_{\lambda,a} \times e(\psi) = \begin{cases} 0 & [\lambda,\psi] = a+1 \\ 2^{-3}[e(\psi\lambda) - e(\psi) - e_{\lambda,\psi}] & [\lambda,\psi] = a. \end{cases}$$

Table 5.11. Inner Products of Idempotents

See the basic inner products in Section 2. We also need $(f(\varphi), f(\psi))$ from (4.15).

$$(e_{\lambda,a}, e_{\mu,b}) = 2^{-9}\langle\lambda,\mu\rangle^2 + 2^{-5}(-1)^{a+b}\delta_{\lambda,\mu} = \begin{cases} 2^{-4} & \lambda = \mu \\ 0 & \lambda\mu \text{ singular} \\ 2^{-7} & \lambda\mu \text{ nonsingular} \end{cases}.$$

$$(e_{\lambda,a}, e(\varphi)) = \\ 2^{-8} + 2^{-8}(-1)^a\varphi(\alpha) = \begin{cases} 2^{-7} & \text{if } (-1)^a\varphi(\alpha) = 1, \ i.e., \ a + [\varphi,\alpha] = 0 \\ 0 & \text{if } (-1)^a\varphi(\alpha) = -1 \ i.e., \ a + [\varphi,\alpha] = 1 \end{cases}.$$

$$(e(\varphi), e(\psi)) = 2^{-8} + 2^{-12}\begin{cases} 240 & \varphi = \psi \\ -16 & \varphi\psi \text{ singular} \\ 16 & \varphi\psi \text{ nonsingular} \end{cases} = \begin{cases} 2^{-4} \\ 0 \\ 2^{-7} \end{cases}.$$

6. Idempotents and Involutions.

Notation 6.1. The polynomial $p(t) := \frac{32}{3}t^2 - \frac{32}{3} + 1$ takes values $p(0) = p(1) = 1$ and $p(\frac{1}{4}) = -1$. For an idempotent e such that $ad(e)$ is semisimple with eigenvalues $0, \frac{1}{4}$ and 1, we define $t(e) := p(ad(e))$, an involution which is 1 on the $0-$ and 1-eigenspaces and is -1 on the $\frac{1}{4}$-eigenspace. Let $E_\pm = E_\pm(e) = E_\pm(t(e))$ denote the ± 1 eigenspace of this involution.

The main results of this section are the following.

Theorem 6.2. *For a quooty or tooty idempotent, e, $t(e)$ is an automorphism of V^F.*

This follows from the theory in [Miy] and (5.2). In this section, we shall verify this directly on the algebra V_2^F only, for the $e_\lambda^\pm$ and $e(\varphi)$ and prove that these elements are all the idempotents whose doubles are conformal vectors of conformal weight $\frac{1}{2}$. See (6.5) and (6.6).

Theorem 6.3. *The subgroup of $Aut(V^F)$ generated by all $t(e)$ as in (6.2) is isomorphic to $O^+(10,2)$.*

The Miyamoto theory proves that the $t(e)$ are in $Aut(V^F)$. It turns out that the group they generate restricts faithfully to V_2^F is faithful, and there we can identify it.

Theorem 6.4. *The group generated by the $t(e_\lambda^\pm)$ is isomorphic to the maximal 2-local subgroup of $O^+(10,2)$ of shape $2^8{:}O^+(8,2)$. The normal subgroup of order 2^8 is generated by all $t(e_\lambda^+)t(e_\lambda^-)$ and acts regularly on the set of weighty idempotents. A complement to this normal subgroup is the stabilizer of any $e(\varphi)$, for example, the stabilizer of $e(1)$ (1 means the trivial character) is generated by all $t(e_{\lambda,1})$. Such a complement is isomorphic to the Weyl group of type E_8, modulo its center.*

To verify that the involution $t(e)$ is an automorphism of V_2^F, it suffices to check that $E_+E_+ + E_-E_- \le E_+$ and $E_+E_- \le E_-$.

Proposition 6.5. *If e is quooty, $t(e)$ is an automorphism of V_2^F.*

Proof. This is straightforward with (6.4) and (5.4).

Proposition 6.6. *If e is tooty, $t(e)$ is an automorphism of V_2^F.*

Proof. This is harder. We use (5.1), (6.4), (5.8) and (5.11). It is easy to verify that $E^+ \times E^+ \le E^+$. To prove $E^- \times E^- \le E^+$, we verify that $(E^- \times E^-, E^-) = 0$ (this suffices since the eigenspaces are nonsingular and pairwise orthogonal); the verification is a straightforward checking of cases. To prove that $E^- \times E^+ \le E^-$, we use the previous result, commutativity of the product and associativity of the form.

Table 6.7. (i) The action of $t(e_{\lambda,a})$ on $\mathfrak{Q}\mathfrak{T}\mathfrak{I}$:
fixed are

$$e_{\mu,b} \text{ if } \langle \mu, \lambda \rangle = 0 \text{ or } \pm 4;\ e(\varphi) \text{ if } [\lambda, \varphi] = 0;$$

interchanged are

$e_{\mu,b}$ and $e_{\lambda+\mu,a+b+1}$ if $\langle\lambda,\mu\rangle = -2$; $e(\varphi)$ and $e(\varphi\lambda)$ if $[\lambda,\varphi] = 1$.

(ii) The action of $t(e(\varphi))$ on $\mathfrak{QTJ}$:
fixed are

all $e'_{\lambda,\varphi}$ and all $e(\psi)$ with $\varphi = \psi$ or $\varphi\psi$ singular;

interchanged are

all $e(\varphi\lambda)$ and $e_{\lambda,\varphi}$ with λ a quoot.

Proof. For an involution t to interchange vectors x and y in characteristic not 2, it is necessary and sufficient that t fix $x+y$ and negate $x-y$. The following is a useful observation: since the $+1$ eigenspace for $t = t(e)$ is the sum of the 0-eigenspace and the $\frac{1}{4}$-eigenspace for $ad(e)$, a vector u is fixed by $t(e)$ iff $e \times u$ is in the $\frac{1}{4}$-eigenspace. Another useful observation is that if $x - y$ is negated, then $(x-y)^2$ is fixed. The proof of (i) and (ii) is an exercise in checking cases.

The identification of G, the group generated by all such $t(e)$, $e \in \mathfrak{QTJ}$, is based on a suitable identification of this set of involutions with nonsingular points in $\mathbb{F}_2^{10}$ with a maximal index nonsingular quadratic form.

Notation 6.8. Let $T := \mathbb{F}_2^{10}$ have a quadratic form q of maximal Witt index. Decompose $T = U \perp W$, with $dim(U) = 2$, $dim(W) = 8$, both of plus type. Let $U = \{0, r, s, f\}$, where $q(f) = 1, q(r) = q(s) = 0$. Identify W with $Hom(L, \{\pm 1\})$. For $x \in V$ nonsingular, write $x = p + y$, for $p \in U, y \in W$. If $p = 0$, correspond x to $e_{y,1}$. If $p = r$, correspond x to $e_{y,0}$. If $p \in \{s, f\}$, correspond $e(y)$ to x. This correspondence is G-equivariant; use (6.7).

So, we have a map of G onto $O^+(10,2)$ by restriction to V_2^F. Its kernel fixes all of our idempotents, which span V_2^F. By Corollary 6.2 of [DGH], this kernel is trivial. So, $G \cong O^+(10,2)$ and (6.3) is proven.

Proposition 6.9. *G acts irreducibly on $\mathbb{I}^\perp$ (dimension 155).*

Proof. This follows from the character table of $\Omega^+(10,2)$, but we can give an elementary proof.

(1) The subgroup H of (6.2) has an irreducible constituent P of dimension 120 with monomial basis v_α, $\alpha \in L_2$;

(2) the squares of the v_α generate the 36-dimensional orthogonal complement, $P^\perp$. The action fixes $\mathbb{I}$ and the action on the 35-dimensional space $P^\perp \cap \mathbb{I}^\perp$ is nontrivial, hence irreducible (the subgroup $O_2(H) \cong 2^8$ acts trivially and the quotient $H/O_2(H) \cong O^+(8,2)$ acts transitively on the spanning set of 120 elements $v_\alpha^2 = \alpha^2$, so acts faithfully. Now, the subgroup $2^6{:}O^+(6,2) \cong 2^6{:}Sym_8$ has smallest faithful irreducible degrees 28 and 35; if H is reducible on $P^\perp \cap \mathbb{I}^\perp$, then 28 occurs and H has an irreducible R of dimension d, $28 \le d \le 34$ and so $P^\perp \cap R^\perp$ is a trivial module of dimension $36 - d \ge 2$. This is impossible since $P^\perp$ is an H-constituent of a transitive permutation module of degree 120, contradiction).

(3) We now have $V_2^F = 1 + 35 + 120$ as a decomposition into H-irreducibles. But each $t(e(\varphi))$ fixes $\mathbb{I}$ and does not fix the 120-dimensional constitutent, whence irreduciblity of G on $\mathbb{I}^\perp$.

Theorem 6.10. $Aut(V^F) = G \cong O^+(10,2)$.

Proof. Set $A := Aut(V^F)$. We quote Theorem (6.13) of [Miy], which says that if $\mathfrak{X}$ is the set of conformal vectors of central charge $\frac{1}{2}$, then $|t(x)t(y)| \in \{1,2,3\}$. So, if $\mathfrak{X}$ is a conjugacy class, it is a set of 3-transpositions. If it is not a conjugacy class, we have a nontrivial central product decomposition of $\langle \mathfrak{X} \rangle$, which is clearly impossible since A acts faithfully and G acts irreducibly on V_2^F. Now, the classification of groups generated by a a class of 3-transpostions [Fi69][Fi71] may be invoked to identify A. It is a fairly straightforward exercise to eliminate any 3-transposition group which properly contains $O^+(10,2)$.

7. A related subVOA of $V^\natural$.

The VOA defined in [FLM], denoted $V^\natural$, has the monster as its automorphism group. One of the parabolics, $P \cong 2^{10+16}\Omega^+(10,2)$, acts on the subVOA V' of fixed points of $O_2(P)$; the degree 2 part V_2' contains V_2^F. In fact, V_2' is isomorphic to the direct sum of algebras V_2^F (with $\times$) and $\mathbb{C}$. The proper subVOA V'' of V' generated by the V_2^F-part is isomorphic to V^F (this is so because we can see our $L = Q^{[E]}$ embedded in the Leech lattice, as the fixed point sublattice of an involution). This subVOA V'' contains idempotents given by formulas like ours for quooty and tooty ones, but these idempotents have $\frac{1}{16}$ in their spectrum on $V^\natural$, so the involutions associated to them by the Miyamoto theory act trivially on V'. The involutory automorphisms of V' given by our forumlas in Section 6 do not extend to automorphisms of $V^\natural$ since otherwise the stabilizer of this subVOA in $I\!M$, the monster, would induce $O^+(10,2)$ on it, contrary to the above structure of the maximal 2-local P; we mention that the maximal 2-locals have been classifed [Mei].

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