Midwest cousins of Barnes-Wall lattices

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Abstract

Given a rational lattice and suitable set of linear transformations, we construct a cousin lattice. Sufficient conditions are given for integrality, evenness and unimodularity. When the input is a Barnes-Wall lattice, we get multi-parameter series of cousins. There is a subseries consisting of unimodular lattices which have ranks $2^{d-1} \pm 2^{d-k-1}$, for odd integers $d \geq 3$ and integers $k = 1, 2, \ldots, \frac{d-1}{2}$. Their minimum norms are moderately high: $2^{\left\lfloor \frac{d}{2} \right\rfloor - 1}$.

**Keywords:** even integral lattice, minimum norm, Barnes-Wall, finite group, 2/4 generation, commutator density
1 Introduction

In this article, lattice means a finite rank free abelian group with rational-valued positive definite symmetric bilinear form.

We develop a general lattice construction method which is inspired by finite group theory. We call it a \textit{midwest procedure} because many significant developments in finite group theory took place in the American midwest.
during the late twentieth century, mainly in Illinois, Indiana, Michigan, Ohio and Wisconsin.

The idea is to start with a lattice $L$ and take a finite subgroup $F$ of $O(\mathbb{Q} \otimes L)$. In the rational span of $F$ in $\text{End}(\mathbb{Q} \otimes L)$, we take an element $h$. We define a new lattice, $L'$, in some way using $L$ and $h$, for example $L \cap \text{Ker}(h)$, $L^* \cap \text{Ker}(h)$, $Lh$, . . . , or sums of such things. After finitely many repetitions of this procedure, the sequence $L, L', . . .$ arrives at a new lattice, which is called a midwest cousin of $L$.

We specialize to the dimension $2^d$ Barnes-Wall lattices $BW_{2^d}$ and the Bolt-Room-Wall groups $BRW^+(2^d)$, of shape $2^{1+2d}\Omega^+(2d, 2)$, which are the full isometry groups of $BW_{2^d}$ if $d \neq 3$. The sophisticated groups $BRW^+(2^d)$ help us manage the linear algebra and combinatorics. We create multi-parameter series of cousin lattices, called the first cousins of the Barnes-Wall lattices. The dimension of a member is a number of the form $2^{d-1} \pm 2^{d-k-1}$, for some $k \in \{1, 2, \ldots, \lfloor \frac{d}{2} \rfloor\}$. The auxiliary finite isometry groups $F_i$ are cyclic groups of orders 2 and 4. When $d$ is odd and $d - 2k \geq 3$, the minimum norms are $2^{\lfloor \frac{d}{2} \rfloor - 1}$ and the lattices are even and unimodular. We include a partial analysis of minimal vectors.

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1.1 Conventions and List of Notations

Group elements and endomorphisms usually act on the right. Table 1 summarizes notations. For background, we recommend [10, 13, 14, 12].

2 Background

Standard properties of Reed-Muller binary codes [19, 18] and the Barnes-Wall lattices [1, 3, 13, 14] will be used intensely. For convenience, we review them here.
<table>
<thead>
<tr>
<th>Notation</th>
<th>Summary</th>
<th>Comments</th>
</tr>
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<tbody>
<tr>
<td>$BRW^+(2^d)$</td>
<td>the Bolt-Room-Wall group, $2_1^{1+2d}\Omega^+(2d, 2)$</td>
<td></td>
</tr>
<tr>
<td>$BW_{2^d}$</td>
<td>the Barnes-Wall lattice of rank $2^d$</td>
<td></td>
</tr>
<tr>
<td>BW-level</td>
<td></td>
<td>(2.6)</td>
</tr>
<tr>
<td>commutator density</td>
<td></td>
<td>(2.22)</td>
</tr>
<tr>
<td>CIN($D$)</td>
<td>category of modules for dihedral group $D$ where central involution acts as negative 1</td>
<td></td>
</tr>
<tr>
<td>clear, grey</td>
<td>elements of $G_d$ with trace nonzero, zero</td>
<td></td>
</tr>
<tr>
<td>core</td>
<td>$S_1 \cap \cdots \cap S_r$ as in cubi sum (below)</td>
<td>(2.15)</td>
</tr>
<tr>
<td>cubi sum</td>
<td>$S_1 + \cdots + S_r$, $S_i$ affine codimension 2</td>
<td>(2.14)</td>
</tr>
<tr>
<td>defect</td>
<td>invariant of an involution in $BRW^+(2^d)$</td>
<td>(2.13)</td>
</tr>
<tr>
<td>$\varepsilon_S$</td>
<td>linear map taking $v_i$ to $v_i, -v_i$, as $i \notin S, i \in S$</td>
<td></td>
</tr>
<tr>
<td>fourvolution</td>
<td>an isometry of order 4 whose square is $-1$</td>
<td></td>
</tr>
<tr>
<td>frame, lower frame</td>
<td></td>
<td>(2.20)</td>
</tr>
<tr>
<td>$G, G_d$</td>
<td>$BRW^+(2^d)$, a subgroup of $O(BW_{2^d})$</td>
<td></td>
</tr>
<tr>
<td>Jordan number</td>
<td>$k^{th}$ layer</td>
<td>(3.4)</td>
</tr>
<tr>
<td>$L(k)/L(k-1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>level</td>
<td>least $\ell$ so that $2^\ell x$ has integer coordinates</td>
<td>(2.6)</td>
</tr>
<tr>
<td>level sublattice</td>
<td></td>
<td>(5.1)</td>
</tr>
<tr>
<td>$L^\varepsilon(t)$</td>
<td>eigenlattice for involution $t$</td>
<td></td>
</tr>
<tr>
<td>$k^{th}$ level, $L(k)$</td>
<td>the set of lattice elements of level at most $k$</td>
<td></td>
</tr>
<tr>
<td>long codeword,</td>
<td>$RM(2, d)$ codeword of weight more than $2^{d-1}$</td>
<td></td>
</tr>
<tr>
<td>$MC(L, t, f, \varepsilon)$</td>
<td>a cousin lattice</td>
<td></td>
</tr>
<tr>
<td>$MC_1(d, k, \varepsilon)$</td>
<td>a cousin lattice</td>
<td></td>
</tr>
<tr>
<td>midset</td>
<td>codeword in $\mathbb{F}_2^n$ of weight $\frac{1}{2}n$</td>
<td>(2.13)</td>
</tr>
<tr>
<td>midword</td>
<td>codeword in $\mathbb{F}_2^n$ of weight $\frac{1}{2}n$</td>
<td></td>
</tr>
<tr>
<td>$\mu(L)$</td>
<td>the minimum norm in the lattice $L$</td>
<td></td>
</tr>
</tbody>
</table>
Table 2: List of Notations, Part 2

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(L)$</td>
<td>isometry group of the quadratic space $L</td>
</tr>
<tr>
<td>$O_p(X)$</td>
<td>the largest normal subgroup of the group $X$ of order divisible by $p$</td>
</tr>
<tr>
<td>$O_{p'}(X)$</td>
<td>the largest normal subgroup of the group $X$ of order prime to $p$</td>
</tr>
<tr>
<td>$\mathcal{P}(X)$</td>
<td>the power set of the set $X</td>
</tr>
<tr>
<td>$\mathcal{PE}(X)$</td>
<td>even subsets, codimension 1 in $\mathcal{P}(X)$</td>
</tr>
<tr>
<td>$\text{quotient code}$</td>
<td></td>
</tr>
<tr>
<td>$R, R_d$</td>
<td>$O_2(G_d)$</td>
</tr>
<tr>
<td>$RM(k,d)$</td>
<td>the Reed-Muller code of length $2^d$, spanned by subspaces of codimension $k$</td>
</tr>
<tr>
<td>$\text{RM-level}$</td>
<td></td>
</tr>
<tr>
<td>$sBW, ssBW$</td>
<td>scaled, suitably scaled Barnes-Wall lattice</td>
</tr>
<tr>
<td>short codeword,</td>
<td>codeword in $\mathbb{R}^n$ of weight $&lt; \frac{1}{2}n$</td>
</tr>
<tr>
<td>short involution</td>
<td></td>
</tr>
<tr>
<td>split, nonsplit</td>
<td>involution of $G_d$ which centralizes, does not centralize, a lower elementary abelian $2^{d+1}$</td>
</tr>
<tr>
<td>standard generators</td>
<td>certain set of $2^{-m}v_A$ in $BW_{2^d}$</td>
</tr>
<tr>
<td>$t$</td>
<td>$\varepsilon_A$, a diagonal involution in $BRW^+(2^d)$</td>
</tr>
<tr>
<td>$\tau_\omega$</td>
<td>translation by element $\omega \in \Omega$</td>
</tr>
<tr>
<td>$\tau(\text{core}(Z))$</td>
<td>the group ${\tau_c</td>
</tr>
<tr>
<td>$Tel(L,E)$</td>
<td>total eigenlattice of group $E$ on lattice $L</td>
</tr>
<tr>
<td>$Tel(L,t)$</td>
<td>total eigenlattice of involution $t$ on lattice $L</td>
</tr>
<tr>
<td>$\text{top(x)}$</td>
<td>part of vector $x$ representing the highest power of 2 in denominator</td>
</tr>
<tr>
<td>$\text{top closure}$</td>
<td>$\text{top}(x)$ is in the lattice if $x$ is in the lattice</td>
</tr>
<tr>
<td>$V^\varepsilon(t)$</td>
<td>$\mathbb{Q} \otimes L^\varepsilon(t)$</td>
</tr>
<tr>
<td>$v_i, v_X \in \mathbb{R}^\Omega$</td>
<td>$(v_i, v_j) = 2^{\frac{d+1}{2}}\delta_{ij}$; $v_X := \sum_{i \in X} v_i$</td>
</tr>
<tr>
<td>$Z, \Omega \in RM(2,d)$</td>
<td>weight $2^{d-1} \pm 2^{d-k-1}$ codewords</td>
</tr>
<tr>
<td>$2/4$ generation</td>
<td></td>
</tr>
<tr>
<td>$\Omega, \Omega_d$</td>
<td>index set for orthogonal basis of $\mathbb{R}^{2^d}$</td>
</tr>
</tbody>
</table>
2.1 Review of Reed-Muller codes

Notation 2.1. For integers \( d \geq 1 \) and \( k \in \{0, 1, \ldots, d\} \), there is defined a Reed-Muller binary code \( RM(k, d) \) of length \( 2^d \). We use \( \Omega = \Omega_d \), a copy of affine space \( \mathbb{F}_2^d \), as indices. A binary vector may be interpreted as an \( \mathbb{F}_2 \)-valued function of its index set \( \mathbb{F}_2^d \), or as a subset of the index set (the support of the previous function). Addition is the boolean sum. The Reed-Muller code \( RM(k, d) \) is spanned by the vectors which are the characteristic functions of affine subspaces of codimension at most \( k \) (or, in the power set interpretation \( \mathbb{F}_2^\Omega \), as the actual affine subspaces). For all \( p \leq -1 \), \( RM(p, d) := 0 \).

We mention a few facts for use in this article.

Proposition 2.2. \( 0 = RM(-1, d) \leq RM(0, d) \leq RM(1, d) \leq RM(2, d) \leq \cdots \leq RM(d-1, d) = PE(\Omega) \leq RM(d, d) = P(\Omega) \). For \( k \geq 0 \), \( \dim(RM(k, d)) = \sum_{i=0}^k \binom{d}{k} \).

Proposition 2.3. For \( d \geq 1 \) and for \( i = 0, 1, 2, \ldots, d-1 \), \( RM(i, d) \perp = RM(d-1-i, d) \).

Lemma 2.4. (i) In \( RM(k, d) \), the minimum weight is \( 2^{d-k} \) and the codewords of minimum weight are the affine subspaces of codimension \( k \);

(ii) In \( RM(d-2, d) \), the second-smallest weight is 6 and it is achieved by subsets of the form \( P + Q \), where \( P, Q \) are affine 2-spaces such that \( P \cap Q \) is a 1-set.

Proof. (i) is well-known; see [17], Theorem 3, p. 375 and Theorem 8, p. 380.

(ii) Let \( S \) be a weight 6 codeword and take \( T \subset S \) a 3-set. There is a unique affine 2-space \( P \) so that \( T \subseteq P \). Then \( P \cap S \) is \( T \) or \( P \). If \( P \cap S \) were \( P \), then \( P \subseteq S \), which would imply that \( S + P \) is a 2-set in \( RM(d-2, d) \), whose minimum weight is 4, a contradiction. Therefore \( P \cap S = T \). Consequently, \( Q := P + S \). Then \( |Q| = 4 \), whence \( Q \) is an affine 2-space by (i). Therefore, \( S = P + Q \), as required. \( \square \)

Lemma 2.5. If \( S \in RM(d-i, d) \), \( T \in RM(d-j, d) \), then \( S \cap T \in RM(d-i-j, d) \)

Proof. This follows from the fact that \( RM(d-k, d) \) is spanned by all \( k \)-dimensional affine subspaces. \( \square \)
**Definition 2.6.** For $A \in \mathcal{P}(\Omega)$, we define the $BW$-level of $A$ to be $\max\{m \geq 0 | A \in RM(d - 2m, d)\}$ and the $RM$-level of $A$ to be $\max\{i | A \in RM(d - i, d)\}$. We abbreviate these terms by $BW - level(A)$ and $RM - level(A)$, respectively. We extend the concept of level to elements of $BW_{2^d}$ by using the notation (5.3) with respect to the basis $v_i$ of (2.1).

**Remark 2.7.** If $i = RM - level(A)$, then the elements of $A + RM(d - i - 1, d)$ have RM-level $i$. If $m = BW - level(A)$, then the elements of $A + RM(d - 2m - 2, d)$ have RM-level $m$.

**Proposition 2.8.** Suppose that $\tau$ is a translation in $AGL(d, 2)$. Then

(i) $RM(j, d)(\tau - 1) \leq RM(j - 1, d)$;

(ii) The image of $\tau - 1$ is the set of all $\tau$-invariant codewords. Also, $\mathcal{P}(\Omega)$ is a free $\mathbb{F}_2[(\tau)]$-module.

(iii) If $x \in Ker(\tau - 1) = Im(\tau - 1)$ and $x \in RM(d - k, d)$, there exists $y \in RM(d - k + 1, d)$ so that $x = y(\tau - 1)$.

**Proof.** (i) The first part is obvious since $RM(j, d)$ is spanned by affine subspaces $S$ of codimension $j$, and $S + S\tau$ is either empty or is a $(j + 1)$-dimensional affine subspace.

(ii) Let $\Gamma$ be a subspace of codimension 1 in $\Omega$ which is transverse to $\tau$ (i.e., if $\tau = \tau_c$, then $\Gamma$ should be transverse to $\{0, c\}$). Then $\mathcal{P}(\Gamma)$ and $\mathcal{P}(\Gamma + \Omega)$ are affine subspaces of $\mathcal{P}(\Omega)$ which are interchanged by $\tau$, intersect in 0 and have dimension $2^{d-1}$. Therefore, $\mathcal{P}(\Gamma)$ is a basis of $\mathcal{P}(\Omega)$, considered as a free $\mathbb{F}_2[(\tau)]$-module.

(iii) Since $\mathcal{P}(\Omega)$ is a free module for $\mathbb{F}_2[(\tau)]$ (by (ii)), $Ker(\tau - 1) = Im(\tau - 1)$. Assume that $c$ is a $\tau$-invariant codeword in $RM(k, d)$. Since $\tau$ is an involution, $c$ is an even set, whence $k \leq d - 1$. Let $h$ be an affine hyperplane which is transverse to every $\tau$-invariant 1-space. Then $c \cap h \in RM(k + 1, d)$ and $c = (c \cap h)(\tau - 1)$. □

**Lemma 2.9.** Let $X$ be a subset of $\Omega$. Then

(i) if $|X|$ is even, $X(\tau - 1)$ is in $RM(d - 2, d)$; and

(ii) if $|X|$ is odd, there is $Q$, a 1-space invariant under $\tau$, such that $X(\tau - 1)$ is in $Q + RM(d - 2, d)$.

(iii) In (ii), if $Q, Q'$ are 1-spaces such that $X(\tau - 1)$ is in $Q + RM(d - 2, d) = Q' + RM(d - 2, d)$, then $Q'$ is a translate of $Q$ and both are $\tau$-invariant.

**Proof.** To prove (i), use (2.8)(i). Next, (ii) follow easily from the case $|X| = 1$. For (iii), we may assume $X$ is a 1-set. First notice that since
Q + Q' ∈ RM(d − 2, d), whose minimal weight codewords are affine 2-spaces, Q' is a translate of Q. One is τ-invariant if and only if the other one is. On the other hand, there exists some 1-space Q" which is τ-invariant and which satisfies X(τ − 1) ∈ Q" + RM(d − 2, d) (just take Q" = \{x, xτ\}, for any x ∈ X, and use (i),(ii)). Therefore, both Q and Q' are τ-invariant. □

Lemma 2.10. Let W be a 2-set in RM(d − 2, d) which is not fixed by the translation τ. Then W + Wτ is a 2-space.

Proof. Clearly W + Wτ has cardinality 4 and is the union of two translates of a 1-space. □

Definition 2.11. Suppose that Γ is a subspace of Ω. Let P(Ω, Γ) be the members of P(Ω) which are unions of cosets of Γ. Then members of P(Ω, Γ) may be interpreted as subsets of the quotient vector space Ω/Γ and so we have an isomorphism P(Ω, Γ) → P(Ω/Γ). This may be interpreted as an isomorphism of a subspace of binary vectors of length |Ω| with the full space of binary vectors of length |Ω/Γ|.

2.2 Review of RM(1, d) and RM(2, d)

These two Reed-Muller codes are widely used and have been analyzed a great deal. The code RM(1, d) has dimension d + 1 and is easy to visualize: it consists of 0, Ω and the set of all affine hyperplanes.

We need more terminology to discuss sets in the more complex code RM(2, d). The article [12] contains references for the rest of this subsection.

Proposition 2.12. The sets in RM(2, d) have cardinalities from the list \{0, 2d−1 − 2d−k−1, 2d−1, 2d−1 + 2d−k−1, 2d|k = 1, 2, \ldots, \lfloor \frac{d}{2} \rfloor\}. For a fixed cardinality other than 2d−1, the codewords of that cardinality form an orbit under the action of AGL(d, 2). For cardinality 2d−1, there are \(1 + \lfloor \frac{d}{2} \rfloor\) orbits.

Definition 2.13. Given a codeword c ∈ RM(2, d), there is at most one integer k ∈ \{1, 2, \ldots, \frac{d}{2}\} such that the coset c + RM(2, d) contains a codeword of weight 2d−1 − 2d−k−1. If there is such a k, we say c has defect k. If there is no such k, we say that c has defect 0. We say that c is short if it has cardinality less than 2d−1, long if it has cardinality greater than 2d−1 and otherwise we say c is a midset or a midword. [12]
Definition 2.14. A sum \( S_1 + \cdots + S_k \) of \( k > 0 \) affine codimension 2 subspaces whose intersection is nonempty, is called a *cubi sum* if its cardinality is \( 2^{d-1} - 2^{d-k-1} \). A short defect \( k \) codeword \( c \) may be written as a cubi sum. We define the *core* of a cubi sum to be the intersection of the \( k \) summands. It depends only on \( c \) and not on the particular cubi sum for \( c \).

Definition 2.15. When \( A \in R(2,d) \) is short, we define \( \text{core}(A) \) as in [12]. When \( A \) is long, we define \( \text{core}(A) \) to be the core of the complement of \( A \). When \( A \) is a midset, we do not define the core (in the coset \( A + RM(1,d) \), the cores which occur for nonmidsets form an orbit under the translation group).

Lemma 2.16. Let \( A, B \in RM(2,d) \) and suppose that \( 0 \neq A < B \neq \mathbb{F}_2^d \). Let \( X + \Omega \) denote the complement of the subset \( X \) of \( \Omega = \mathbb{F}_2^d \). Then one of the following holds:

(i) \( A \) is a codimension 2 subspace and \( B \) is a midset; or \( B \) is a codimension 2 subspace and \( A + \Omega \) is a midset.

Furthermore, (i) happens for affine hyperplanes \( B \) for any \( d \geq 3 \), and for nonaffine midsets \( B \) exactly when \( B \) has defect 1 and \( d \geq 3 \), respectively.

(ii) \( A \) is short and \( B \) is long, of respective cardinalities \( 2^{d-1} - 2^{d-k-1} \), \( 2^{d-1} + 2^{d-r-1} \), where \((k,r) = (1,1), (1,2), (2,1) \) or \((2,2)\). We summarize:

\[
\begin{array}{|c|c|c|c|}
\hline
(k,r) & |A| & |B| & |A + B| \\
\hline
(1,1) & 2^{d-1} - 2^{d-2} = 2^{d-2} & 2^{d-1} + 2^{d-2} = 2^{d-2}3 & 2^{d-1} \\
(2,1) & 2^{d-1} - 2^{d-3} = 2^{d-3}3 & 2^{d-1} + 2^{d-2} = 2^{d-2}3 & 2^{d-3}3 \\
(1,2) & 2^{d-1} - 2^{d-2} = 2^{d-2} & 2^{d-1} + 2^{d-3} = 2^{d-3}5 & 2^{d-3}3 \\
(2,2) & 2^{d-1} - 2^{d-3} = 2^{d-3}3 & 2^{d-1} + 2^{d-3} = 2^{d-3}5 & 2^{d-2} \\
\hline
\end{array}
\]

Note that cases \((1,2)\) and \((2,2)\) are dual in the sense that \( A \) and \( A+B \) may be interchanged. Note that the case \((1,1)\) corresponds to (i) for the midset \( A + B \) containing \( B + \Omega \). Note also that \( A \) in case \((1,2)\) and \( A + B \) in case \((2,2)\) are codimension 2 affine spaces.

Proof. [12] □

2.3 Review of PO2\(^d\)-theory and Barnes-Wall lattices

The Reed-Muller codes can be used to construct Barnes-Wall lattices [1], [3]. Alternatively, they may be deduced from existence of Barnes-Wall lattices [13], [14].
Notation 2.17. The Barnes-Wall lattice $BW_{2d}$ in rank $2^d$, $d \geq 2$, is an even lattice whose isometry group contains $G_d \cong 2_1^{1+2d}\Omega^+(2d,2)$. This is the full isometry group when $d \neq 3$. These lattices are scaled so as to make $BW_{2d}$ unimodular when $d$ is odd and to make the discriminant group elementary abelian of rank $2^{d-1}$ when $d$ is even. Finally, define $R_d := O_2(G_d) \cong 2_1^{1+2d}$.

The Barnes-Wall lattices can actually be defined as the unique (up to scaling) rational lattice invariant under $2_1^{1+2d}\Omega^+(2d,2)$.

The earliest constructions involved building up a square lattice in layers using the Reed-Muller codes [1], [3]. They may be also built by induction in a way which avoids some of the counting and linear algebra work [13].

The lattices of [1] provided an example (perhaps the first) of a family of unimodular lattices where the minimum norms went to infinity. The rate is roughly as square root of the dimension.

The much-studied Leech lattice involves $BW_{24}$ as a sublattice.

Definition 2.18. For $BW_{2d}$, there are two standard generating sets (as abelian groups). We start with the a set $\{v_i | i \in \Omega\}$ of vectors in $BW_{2d}$. As in (2.1), $\Omega = \mathbb{F}_2^d$. We often use the maps $\varepsilon_S$, which take $v_i$ to $-v_i$ if $i \in S$ and to $v_i$ if $i \not\in S$. This map is in $G_d$ if and only if $S \in RM(2,d)$ and is in $R_d$ if and only if $S \in RM(1,d)$ (2.17).

The (first) standard generating set is the set of of vectors of the form $\frac{1}{2^m} v_A$, where $m$ is a nonnegative integer, $0 \leq m \leq 2^{\lfloor \frac{d}{2} \rfloor}$, and $A \in RM(d-2m,d)$, for all $m \geq 0$. There is a second standard generating set is all of vectors of the form $\frac{1}{2^m} v_A$, where $m$ is a nonnegative integer and $A$ is an affine $2m$-space in $\Omega$. This is a proper subset of the standard generating set.

Proposition 2.19. (i) The minimal vectors in $BW_{2d}$ are of the form $\frac{1}{2^m} v_A\varepsilon_S$, where $m$ is a nonnegative integer, $0 \leq m \leq 2^{\lfloor \frac{d}{2} \rfloor}$, $A$ is an affine $2m$-space in $\Omega$ and $C \in RM(2,d)$. They have norms $2^{\lfloor \frac{d}{2} \rfloor}$.

(ii) The minimal vectors in $BW_{2d}[-1]$ are of the form $\frac{1}{2^m} v_A\varepsilon_S$, where $m$ is a nonnegative integer, $0 \leq m \leq 2^{\lfloor \frac{d}{2} \rfloor}$, $A$ is an affine $2m-1$-space in $\Omega$ and $C \in RM(2,d)$. They have norms $2^{\lfloor \frac{d}{2} \rfloor-1}$. (The notation $BW_{2d}[-1]$ means $BW_{2d}(f-1)^{-1}$, where $f$ is a lower fourvolution (2.25), (2.27)).

Proof. (i) This is a standard result [3, 13]. (ii) This analogue of (i) about $BW_{2d}[-1]$ may be proved by induction following the ideas of [13]. □
Definition 2.20. Let \( L := BW_{2d} \). A lower frame is a set of \( 2^{d+1} \) minimal vectors of \( L \) which forms an orbit under the action of the normal extraspecial subgroup of order \( 2^{1+2d} \) of \( BRW^+(2^d) \). (A lower frame was called a sultry frame in [12].)

Lemma 2.21. Suppose that we have two sublattices \( M, N \) such that \( E_8 = M + N \) and \( M \cong N \cong \sqrt{2}E_8 \). There exists \( \gamma \in O(E_8) \) which interchanges \( M \) and \( N \).

Proof. This follows from the analogous property of \( O^+(2d,2) \) since \( O(E_8) \) acts on \( E_8 \) mod 2 as \( O^+(8,2) \). \( \square \)

2.4 Review of commutator density

This concept was introduced in [13]. Let \( D \) be an extraspecial 2-group and let \( CIN(D) \) be the category of modules for which the central involution of \( D \) acts as \(-1\). Often, \( D \) is dihedral of order 8.

The basic results are summarized in this section. For a proof, see [14].

Definition 2.22. Let \( E \) be a group, \( S \) a subset of \( E \) and \( M \) a \( \mathbb{Z}[E] \) module. We say that \( S \) is commutator dense (CD) on \( M \) if \( [M,E] = [M,S] \).

Definition 2.23. Let \( D \) be a dihedral group of order 8 and let \( M \) be a \( \mathbb{Z}[D] \)-module. We say that \( M \) has the \( 2/4 \) generation property if for any pair of involutions \( u, v \) which generate \( D \), we have \( M^+(t) + M^+(u) = M \).

Proposition 2.24. Let \( D \) be a dihedral group of order 8 and let \( M \) be a \( \mathbb{Z}[D] \)-module on which the central involution of \( D \) acts as \(-1\). Let \( f \in D \) have order 4. Then on \( M \), \( 2/4 \)-generation and commutator density of \( \{f\} \) are equivalent.

Proof. [14] \( \square \)

Notation 2.25. Suppose that \( D \) is dihedral of order 8 and that \( L \) is in the category \( CIN(D) \). Let \( f \) be an element of order 4 in \( D \) and let \( p \) be an integer. The \( p \)-th twist of \( L \) is the \( D \)-submodule \( L(f-1)^p \) of \( \mathbb{Q} \otimes L \).

Lemma 2.26. Suppose that \( D \) is dihedral of order 8 and that the rational vector space \( V \) is in \( CIN(D) \). Suppose that \( L, M \) are \( D \)-submodules of \( V \) and that \( L, M \) have commutator density. Let \( f \) be an element of order 4 in \( D \). Then, for all integers \( p, q \), \( (L[p] + M[q], D) \) has commutator density.
Proof. Trivially, the sum of two abelian groups in $CIN(D)$ is in $CIN(D)$ and their sum has commutator density if each has commutator density. It suffices to prove that a twist of $L$ has commutator density. Let $u$ and $v$ be generators of $D$. Then $L = L^+(u) \oplus L^+(v)$. Conjugation by $f - 1$ in $GL(Q \otimes L)$ normalizes $D$ and induces on $D$ an outer automorphism. It follows that $L[p] = L(f - 1)^p$ is the direct sum of its fixed points by $(f - 1)^{-p}u(f - 1)^p$ and $(f - 1)^{-p}v(f - 1)^p$. □

Proposition 2.27. Let $f \in R_d$ be a fourvolution. Then $[L, R_d] = L(f - 1)$, i.e., $f$ is commutator dense on the $R_d$-module $L$.

Proof. [13, 14]. □

3 Involutions on Barnes-Wall lattices

Definition 3.1. We use the notations and results of [13] and [12], which are recommended for background. We recall that an involution in $BRW^+(2^d)$ is grey if conjugate to its negative, and is clear otherwise. (Such involutions have been called dirty and clean, respectively [8, 13, 12].

An involution in $BRW^+(2^d)$ is split if it centralizes a maximal elementary abelian subgroup of $R_d$ and is nonsplit otherwise.

For a summary of properties and classification of such involutions, see [12] Appendix: About BRW groups. We have changed some terminology since that article. We mention one often-used result.

Theorem 3.2. (i) If $g \in BRW^+(2^d)$, then the trace of $g$ on the natural $2^d$-dimensional module is 0 if $g$ is grey and is $\pm 2e$ if $g$ is clear, where $2e$ is the dimension of the fixed point subspace for the conjugation action of $g$ on $R_d/Z(R_d)$.

(ii) Suppose that $g$ is an involution. The defect $k$ of $g$ satisfies $e + k = d$. The multiplicities of eigenvalues $\pm 1$ are (up to transposition) $2^{d-1} + 2^{d-k-1}, 2^{d-1} - 2^{d-k-1}$, respectively.

Notation 3.3. Let $A \in RM(2, d)$ be a short codeword of defect $k$. By [12], there are codimension 2 affine spaces $A_1, \cdots, A_k$ so that $A = A_1 + \cdots + A_k$. This is a cubi sum, as discussed in [12]. We let $Z := A + \Omega$ be the complement. Throughout this article, we shall work with involutions of the form $t := \varepsilon_A$. 

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Its trace is \(2^{d-k}\). The affine subspace \(\text{core}(A) = \text{core}(Z) = \cap_i A_i\) is \((d - 2k)\)-dimensional (see [12]). For \(c \in \Omega\), the corresponding translation map is \(\tau_c\). If \(c \in \text{core}(A)\), we call \(\tau_c\), a \textit{core translation}, so when \(\text{core}(A)\) contains the origin, we get a group of translations. Let \(\tau_c\) be a nonidentity core translation. Finally, we take any hyperplane \(H\) which contains no translate of \(c\) and set \(f := \varepsilon_H \tau_c\), a fourvolution which commutes with \(t\).

### 3.1 Involutions on Barnes-Wall lattices mod 2: \(\text{JNo}\)

We begin by studying the Jordan canonical form of involutions on the Barnes-Wall lattice modulo 2. We derive applications to discriminant groups and lattice constructions.

**Definition 3.4.** The \textit{Jordan number} of an involution acting on a finite rank abelian group \(A\) is the number of degree 2 Jordan blocks in its canonical form on \(A/2A\). We write \(\text{JNo}(t)\) or \(\text{JNo}(t, A)\) for the Jordan number of \(t\).

**Lemma 3.5.** On \(BW_{2^d}\), the Jordan number for \(-1\) is 0 and the Jordan number is \(2^{d-2}\) for a lower noncentral involution.

**Proof.** The first statement is obvious. The second follows since \(|BW_{2^d} : Tel(t)| = 2^{2d-2}\) for lower involutions \(t\). See [13]. □

**Notation 3.6.** In this section, the notations of (3.4) will stand for lattices (which often are sBWs) and the involutions will be isometries of them. Let \(L\) be a sBW lattice of rank \(2^d\). If \(t \in \text{O}(L)\) is an involution, as before, we let \(\text{JNo}(t)\) be its Jordan number (3.4). Because of (3.5), we assume that the defect \(k\) is positive, i.e., that the involution is upper. If \(2k < d\), there exists a lower dihedral group in \(C_{G_d}(t)\).

**Theorem (3.15)** is the main goal of this section.

**Lemma 3.7.** If \(t\) is a nonsplit involution, it has full Jordan number, i.e., \(\text{JNo}(t) = 2^{d-1}\).

**Proof.** A nonsplit involution is upper. By [12], there exists a lower dihedral group \(D\) so that \(t\) normalizes \(D\) and effects an outer automorphism on \(D\), say by transposing a set of generators \(u, v\). Using 2/4 generation of \(L\) with respect to \(D\), we get \(L = L^+(u) \oplus L^+(v)\) for a generating pair of involutions \(u, v\) so that \(u^t = v\). Then obviously \(L\) is a free \(\mathbb{Z}(t)\)-module, so we are done. □
Lemma 3.8. If $t$ centralizes a lower dihedral group, $\text{JNo}(t) = \text{JNo}(t') + \text{JNo}(t'')$, where $t', t''$ are defect $k$ involutions on $sBW$ lattices of rank $2^{d-1}$.

Proof. We may choose such a lower dihedral group $D$ to satisfy $D \cap [R, t] = Z(R)$. Use the 2/4 property to get that $t$ preserves each direct summand in $L = L^+(u) \oplus L^+(v)$ for a generating pair of involutions $u, v$ of $D$ (the summands are $sBW$). In the notation of [13], there exists a group $Q \cong 2^{1+2(d-1)}$ in $BRW^+(2^d)$ which acts trivially on $L^-(u)$ and as a lower group on $L^+(u)$. Since the action of $t$ on $R$ has defect $k$, the action of $t$ on $Q$ has defect $k$. We may therefore apply induction to the restriction of $t$ to the summand $L^+(u)$. A similar argument applies to $L^+(v)$. □

Lemma 3.9. When $(d, k) = (2, 1)$ and $t$ is an upper involution, $\text{JNo}(t) = 1$ when $t$ is clear and $\text{JNo}(t) = 2$ when $t$ is grey.

Proof. We refer to [13] for a discussion of involutions in $BRW^+(2^2) \cong W_{F_4}$.

Suppose that the involution is clear. Since its trace is $\pm 2$, we may assume that it is 2, whence $t$ is a reflection. Then the statement is obvious since reflections induce transvections on the lattice mod 2.

For $d = 2$, if an involution is upper and nonsplit, we may quote (3.7). For $d = 2$, if an involution is upper and split, it is clear and we may quote the previous paragraph. □

Lemma 3.10. If $t$ is clear, $\text{JNo}(d, k) \leq 2^{d-1} - 2^{d-k-1}$.

Proof. We may assume that $tr(t) > 0$. Let $h$ be the dimension of fixed points for $t$ on $L/2L$. Then $h + \text{JNo}(t) = 2^d$. Since the 1-eigenlattice for $t$ has rank $2^{d-1} + 2^{d-k-1}$ and is a direct summand of $L$, we have $h \geq 2^{d-1} + 2^{d-k-1}$. □

Lemma 3.11. Suppose that the upper involution $t$ lies in a subgroup $S$ of $G$ of order $2n$, $n$ odd, and that every nonidentity element of $S$ of order dividing $n$ has the same fixed point subspace, of dimension $2e$, on $R/R'$. Assume further that $t$ inverts a nonidentity odd order element of $S$. Then $\text{JNo}(t) \geq 2^{d-1} - \frac{1}{2}(\frac{2^d-2e}{n} + 2e)$.

Proof. Such a group $S$ has a normal subgroup of order $n$. Call it $C$. Then every nonidentity element of $C$ has trace $\pm 2e$ on $L$ (3.2). It follows that the eigenlattice $M$ of $C$-fixed points has rank $\frac{1}{n}(2^d + (n-1)2e) = \frac{1}{n}(2^d - 2e + n2e)$. On the annihilator $N := L \cap M^\perp$, $C$ acts faithfully on every constituent, and since $t$ inverts a nonidentity element of $C$, $N/2N$ is a free $(t)$-module, whence $\text{JNo}(t) \geq \frac{1}{2}\text{rank}(N) = \frac{1}{2}(2^d - \text{rank}(M))$. □
Next, we deal with the situation when $t$ does not centralize a lower dihedral group.

**Lemma 3.12.** We use the hypotheses and notation of (3.11).

(i) Suppose that $d$ is even, $n = 2^d + 1$ and $e = 0$. Then $\text{JNo}(t) \geq 2^{d-1} - 2^{d-1}$. 

(ii) Suppose that $d$ is odd, $n = 2^{d-1} + 1$ and $e = 1$. Then $\text{JNo}(t) \geq 2^{d-1} - 2^{d-1}$. 

**Proof.** Straightforward with (3.11). □

**Lemma 3.13.** Suppose that $m \geq 1$, $2r \geq 4m \geq 4$ and that $u$ is an involution in $\Omega^+(2r, 2)$ with commutator submodule of dimension $2m$ on its natural module $W := \mathbb{F}_2^{4m}$. Assume that $W(u - 1)$ is a totally singular subspace. Let $n = 2^{2m} - 1$.

Then $u$ is in a group $P$ of order $2n$, where $P$ contains a Singer cycle $C$ in a natural $GL(2m, 2)$-subgroup of $\Omega^+(2r, 2)$ (so $C$ is a normal subgroup of $P$). Also $P$ has the property that the nonidentity elements of $C$ have the same fixed point subspace on $\mathbb{F}_2^{2r}$.

**Proof.** Recall properties of the normalizer of a Singer cycle in classical groups, [15]. Without loss, we may assume that $2r = 4m$.

Suppose that we are given pair of maximal totally singular subspaces, $W_1, W_2$ in $W$ such that $W = W_1 \oplus W_2$. Let $H$ be the common stabilizer of $W_1$ and $W_2$. So, $H \cong GL(2m, 2)$. Let $P$ be the subgroup of the normalizer of a Singer cycle in $H$ corresponding to the Singer cycle and the group of field automorphisms of order 2. It has order $2n$ and its involutions invert nonidentity elements of $C$ so have Jordan number $2m$ on $W$. If $u$ is conjugate to such an involution, we are done. There are two conjugacy classes of involutions in $\Omega^+(2m, 2)$ with maximal Jordan number $2m$, which form a single class under the action of $O^+(4m, 2)$ [12]. By conjugacy in $O^+(4m, 2)$, $u$ lies in such a group, $P$. □

**Lemma 3.14.** Suppose that $d \geq 2$ and that $t \in G_d$ has defect $\frac{d}{2}$ or $\frac{d-1}{2}$. Let $R := R_d$. Suppose that $[R, t]$ is elementary abelian. Then $t$ is in a dihedral group as in (3.11).

**Proof.** Let bars indicate images in $G_d/R_d$. Lemma (3.13) implies that $\bar{t}$ is in an appropriate Singer normalizer, $E$. Let $u$ be a conjugate of $t$ in $G$ so that $\bar{t}\bar{u}$ generates $O_2(E)$. There exists $c \in \langle tu \rangle$ which generates a cyclic group of
odd order which maps isomorphically onto $O'_{2}(E)$. Then $\langle t, c \rangle$ satisfies the conclusion. □

Now we prove the main result (3.15).

**Theorem 3.15.** Let $d \geq 2$ and let $t$ be an upper involution in $BRW^+(2^d)$ of defect $k \geq 1$. Then $\text{JNo}(t) = 2^{d-1} - 2^{d-k-1}$ if $t$ is split, and is $2^{d-1}$ if $t$ is nonsplit.

**Proof.** We have $d \geq 2$. Suppose that $[R, t]$ is not elementary abelian. There exists a lower involution $w$ so that $[w, t]$ has order 4. Then on the lower dihedral group $D := \langle w, [w, t] \rangle$, $t$ induces an outer automorphism. Now use (3.7).

We may assume that $t$ is split. So, $[R, t]$ is elementary abelian. If the involution $t$ centralizes a lower dihedral group, the 2/4 generation property (2.23) and induction (3.8) implies the result. Note that the initial cases for induction are discussed in [12].

Assume that the involution $t$ does not centralize a lower dihedral group. Then $R_d/R'_d$ is a free $\mathbb{F}_2[(t)]$-module, $d$ is even and $d = 2k$. We apply (3.14), (3.13) with $r = m = k$, then (3.12) and (3.10). □

### 3.2 Applications to discriminant groups

Knowing $\text{JNo}$ is quite useful. One can get sharp statements about the discriminant group, which might be hard to calculate directly from a definition of the lattice, e.g. by a spanning set.

**Lemma 3.16.** Let the involution $u$ act on the additive abelian group $A$. Then $2A^- \leq [A, u] \leq A^-.$

**Proof.** Clearly, $u$ negates all $a(u - 1)$, so $[A, u] \leq A^-.$ Also, if $a \in A^-$, $2a = a - (-a) = a(1 - u) \in [A, u].$ □

**Corollary 3.17.** Suppose that $tr(t) > 0$. Then $L^-(t) = [L, t]$ for any clear involution $t$.

**Proof.** Since $t$ is an involution, $L^-(t) \geq [L, t]$ (3.16). Since $\text{JNo}(t) = \text{rank}(L^-(t))$ (3.15), the image in $L/2L$ of $[L, t]$ has dimension equal to the rank of $[L, t]$. Therefore, $L^-(t) + 2L = [L, t] + 2L$. Since $[L, t] \leq L^-(t) \leq [L, t] + 2L$, the Dedekind law implies that $L^-(t) \leq [L, t] + (L^-(t) \cap 2L)$. Since $L^-(t)$ is a direct summand of $L$, $L^-(t) \cap 2L = 2L^-(t)$. The latter is contained in $[L, t]$, by (3.16). We conclude that $L^-(t) = [L, t].$ □
Corollary 3.18. Let \( d \geq 2 \). Let \( t \) be a split involution of defect \( k \geq 1 \), and \( \varepsilon = \pm \). Suppose \( \text{tr}(t) > 0 \).

(i) The image of \( L \) in the discriminant group of \( L^\varepsilon(t) \) is 2-elementary abelian of rank \( 2^{d-1} - 2^{d-k-1} \).

(ii) \( L^{-}(t) \leq 2P^{-}(t) \).

(iii) If \( d \) is odd, \( \mathcal{D}(L^{-}(t)) \cong \mathcal{D}(L^{+}(t)) \) is 2-elementary abelian of rank \( 2^{d-1} - 2^{d-k-1} \). In particular, \( L^{-}(t) = 2L^{-}(t)^* = 2P^{-}(L) \).

Proof. (i) The kernel of the natural map \( \pi_{\varepsilon} : L \to \mathcal{D}(L^\varepsilon(t)) \) is \( L^{+}(t) \perp L^{-}(t) \). The cokernel is elementary abelian of rank \( J\text{No}(t) \).

(ii) Use (i) and rank considerations.

(iii) Since \( d \) is odd, unimodularity of \( L \) implies that each \( \pi_{\varepsilon} \) is onto. \( \square \)

4 Midwest procedures

We introduce the first midwest operator here.

Definition 4.1. The midwest cousin (MC) lattices are defined as follows. Let \( L \) be an integral lattice. Let \( t, f \in O(L) \) so that \( t, f \) commute, \( t \) is an involution and \( f \) is a fourvolution. Let \( P^\varepsilon \) be the orthogonal projection to \( V^\varepsilon(t) \). Set \( MC(L, t, f, \varepsilon) := L^\varepsilon + P^\varepsilon(L)[1] = L^\varepsilon + P^\varepsilon(L[1]) \) (see (2.25) for notation \( L[p] \)).

Lemma 4.2. Let \( \varepsilon = \pm \).

(i) The midwest cousin \( MC(L, t, f, \varepsilon) \) is an integral lattice.

(ii) If \( L^\varepsilon(t) \) is doubly even, then \( MC(L, t, f, \varepsilon) \) is an even lattices.

Proof. (i) We verify that \( (x, y) \in \mathbb{Z} \), for \( x, y \in MC(L, t, f, \varepsilon) \). If \( x \) or \( y \) is in \( L^\varepsilon(t) \leq L \), this is clear. Now suppose that \( x = x'(f - 1), y = y'(f - 1) \) for \( x', y' \in P^\varepsilon(L) \). Then \( (x, y) = (x'(f - 1), y'(f - 1)) = 2(x', y') = (x', 2y') \in (L^\varepsilon(t)^*, 2L) \leq (L^\varepsilon(t)^*, \text{Tel}(L, t)) = (L^\varepsilon(t)^*, L^\varepsilon(t)) \leq \mathbb{Z} \).

(ii) We take \( x \in L, y := P^\varepsilon(x) \). Then \( 2y = P^\varepsilon(2x) \in P^\varepsilon(\text{Tel}(L, t)) = L^\varepsilon(t) \) so that \( 2y \in L^\varepsilon(t) \). We have \( (2y, 2y) \in 4\mathbb{Z} \) since by hypothesis, \( L^\varepsilon(t) \) is doubly even. Therefore, \( (y, y) \in \mathbb{Z} \) and so \( y(f - 1) \) has even norm. Since \( L^\varepsilon(t) \) is even, and \( (P^\varepsilon(L), L^\varepsilon(t)) \leq \mathbb{Z} \), it follows that \( MC(L, t, f, \varepsilon) \) is even. \( \square \)

Definition 4.3. The midwest first cousins of the Barnes-Wall lattices are defined as follows. They are the MC lattices with input lattice \( BW_{2^d} \) and a
pair $t, f$ as in (4.1) where $t$ is a clear defect $k$ involution and $f \in C_R(t)$ is a lower fourvolution (3.1). When $k < \frac{d}{2}$, such pairs are unique up to conjugacy in $BRW^+(2^d)$. In this case, we use the briefer notation $MC_1(d, k, \varepsilon)$ for $MC(L, t, f, \varepsilon)$. When $k = \frac{d}{2}$, there are several conjugacy classes of pairs $(t, f)$. One would need additional notation to distinguish these classes [13].

**Remark 4.4.** Suppose that we have two pairs $(t, f)$ and $(t, f')$, where both $f, f'$ are lower involutions which commute with $t$, then the resulting first cousin lattices are the same. The reasons are that $L(f - 1)^p = L(f' - 1)^p$, for all $p$ [13] and the projection maps $P^c$ commute with $f$ and $f'$. In certain commutator calculations, it may be convenient to replace $f - 1$ by $\pm f' + 1$.

### 4.1 Integrality properties of the first cousin lattices

We now specialize to the case of Barnes-Wall lattices.

**Proposition 4.5.** Let $d \geq 2$, $L := BW_{2d}$. We assume that the involution $t$ has defect $k \geq 1$ and that its trace is positive. Then

(i) $\text{rank}(MC_1(d, k, \pm)) = 2^{d-1} \pm 2^{d-k-1}$.

(ii) Let $\varepsilon = \pm$. If $d$ is odd and $d \geq 3$, $MC_1(d, k, \varepsilon)$ is unimodular.

(iii) For $\varepsilon = \pm$, $k \leq \frac{d}{2} - 1$, then $P^c(t)(f - 1)$ is even integral and $L^\varepsilon(t)$ is doubly even (and so $MC_1(d, k, \varepsilon)$ is even).

(iv) $\mu(MC_1(d, k, -)) = \frac{1}{2} \mu(BW_{2d})$.

(v) $\mu(MC_1(d, k, +)) \leq 2^{[\frac{d}{2}]}$.

(vi) If $d = 2k$ or $d = 2k + 1$, $MC_1(d, k, \varepsilon)$ is an odd integral lattice.

**Proof.** For (i), see (3.2).

For (ii), we have that $\frac{1}{2}L^-(t) = P(L)$, which is $L^-(t)^*$ since $L$ is unimodular (3.18)(iii). Consequently, $\mathcal{D}(L^-(t)) \cong 2^{\text{rank}(L^-(t))} = 2^{2^{d-1} - 2^{d-k-1}}$. The lattice $MC_1(d, k, -)$ is between $L^-(t)$ and its dual and corresponds to the image of $f - 1$, where $f$ is a lower fourvolution in $C_R(t)$. In fact, $MC_1(d, k, -) = P^-(L)(f - 1)$. Since $(f - 1)^2 = -2f$ and $|\frac{1}{2}L^-(t) : MC_1(d, k, -)| = |MC_1(d, k, -) : L^-(t)|$, unimodularity follows.

The argument for $\varepsilon = +$ is similar since $\mathcal{D}(L^+(t)) \cong \mathcal{D}(L^-(t))$ as modules for $f - 1$.

(iii) By (4.2), $P^cL(f - 1)$ is integral. We show that it is even under our restrictions on $k$.

Since $k < \frac{d}{2}$, there exists a lower dihedral group $D \leq C_R(t)$ so that $D \cap [R, t] = Z(R)$. If $u, v$ form a generating set of involutions, $L = L^+(u) + L^+(v)$
by 2/4-generation (2.23). The action of $t$ on each summand is clear of defect $k$.

Suppose that $d$ is even. Then $d$ is odd and each summand is $t$-invariant and is isometric to $\sqrt{2}BW_{2d-1}$. By a previous paragraph, the norms of vectors in $P^e(L^+(u))$ and $P^e(L^+(v))$ are integral. Therefore the norms of vectors in $P^e(L^+(u))(f - 1)$ and $P^e(L^+(v))(f - 1)$ are even integral. This suffices to prove (iii) since we have a spanning set of even vectors in an integral lattice.

For (iv), note that $L^-(t)$ contains a minimal vector of $L$ and that $MC_1(d, k, -)$ is the $-1$ twist (2.25) of $L^-(t)$.

(v) This is obvious since $L^+(t)$ contains a minimal vector of $L$.

(vi) Integrality was proved in (4.2)(i).

If $d = 2k$, the vector $v := 2^{-k}v_Z$ is in $P^e(L)$. Its norm is $2^{-2k}2^k(2d-1 + 2d-k-1) = 2d-k+1 + \varepsilon\frac{1}{2}$. The vector $v(f-1)$ is in $MC_1(d, k, +)$ and has odd integer norm.

If $d = 2k+1$, let $H$ be an affine hyperplane which is transverse to $\text{core}(Z)$, which is 1-dimensional. The vector $v := 2^{-k}v_{H\cap Z}$ is in $P^+(L)$ and has norm $2^{-2k}2^k(2d-2 + 2d-k-2) = 2d-k-2\varepsilon\frac{1}{2}$. The vector $v(f-1)$ is in $MC_1(d, k, -)$ and has odd integer norm. To prove the result for $\varepsilon = -$, replace $Z$ by $Z + \Omega$ in the above reasoning.

Suppose that $d = 2k$ is even. Then $2^{-k}v_\Omega \in L$ and $2^{-k}v_Z \in P^+(L)$. Its norm is $2^{-2k}2^k(2d-1 + \varepsilon2d-k-1) = 2k-1 + \frac{1}{2}$. The vector $v(f-1)$ is in $MC_1(d, k, +)$ and has odd integer norm. A similar argument works for $\varepsilon = -$. □

**Remark 4.6.** (i) The unimodular integral lattices $MC_1(5, 2, \pm)$ are not even since their ranks are 20 and 12, which are not multiples of 8. Another proof is (4.5).

(ii) The lattice $MC_1(4, 1, +)$ has rank 12 and determinant $2^8$, though $MC_1(4, 1, -) \cong E_8$ is unimodular. The oddness of $d$ may be necessary for unimodularity of a first cousin.

### 4.2 Minimum norm vectors in $MC_1(d, k, +)$

In this section, we determine that the minimum norm for $MC(d, k, +)$ is $2^{\frac{d-1}{2}}$ (4.9), the same as for $MC_1(d, k, -)$ (4.5). Later, we discuss the forms for low norm vectors in the first few layers (5.1) and study orthogonal decomposability.
Notation 4.7. We let $t$ be a clear involution of defect $k$ and positive trace. We take $t$ to have the form $\varepsilon_{Z'}$, where $Z$ has weight $2^{d-1} + \varepsilon 2^{d-k-1}$ and $Z' = Z$ or $Z + \Omega$, whichever is long. As before, abbreviate $P^\varepsilon$ for the projection to $L^\varepsilon(t)$. We take $\tau := \tau_c$, $f := \varepsilon_H \tau$ and define $\xi := f - 1$, so that $L[k] = L\xi^k$, for all $k$.

Notation 4.8. $\delta := \frac{d-1}{2}$.

Theorem 4.9. We suppose that $d - 2k \geq 3$.

(i) $\mu(MC_1(d,k,\varepsilon)) = 2^{\delta-1}$.

(ii) A vector $v \in MC_1(d,k,\varepsilon)$ is minimal if and only if $v\xi$ is minimal in $L^\varepsilon(t)$ (equivalently, if the support of $v\xi$ is contained in $Z$ and $v\xi$ is a minimal vector of $BW_{2^d}$).

(iii) The minimal vectors of $MC_1(d,k,\varepsilon)$ are in $MC_1(d,k,\varepsilon) \setminus L^\varepsilon(t)$.

Proof. (i) Let $v \in MinVec(MC_1(d,k,\varepsilon))$. Since $v\xi \in L^+(t)$, $(v,v) \geq 2^{\delta-1}$. It suffices to prove that there exists a vector in $MC_1(d,k,\varepsilon)$ of such a norm.

We let $p \geq 1$ and let $A$ be an affine subspace of dimension $2p$ in $\Omega$ which is a translation of a subspace of $core(Z)$ (this is possible since $d - 2k \geq 3$). We also choose $A$ to be transverse to $H$ (this is possible since $2p < d - 2k$) and to be contained in $Z$. Therefore, $A \cap H$ is a $(2p-1)$-dimensional space. The vector $2^{-p}v_{A\cap H}$ is in $MC_1(d,k,\varepsilon)$ and has norm $2^{\delta-1}$.

(ii) Since $\xi$ takes $MC_1(d,k,\varepsilon)$ into $L^+(t)$ and doubles norms, this follows from (i).

(iii) This follows from (ii) since the minimum norm in $L$ is $2^\delta$. $\square$

Corollary 4.10. A minimal vector of $MC_1(d,k,\varepsilon)$ has the form $2^{-m}v_A\varepsilon S$, where $A$ is an affine $(2m-1)$-space, $A \subseteq Z$ and $S \in RM(2,d)$.

Proof. Use (2.19), (4.9). $\square$

Remark 4.11. The description (4.10) of minimal vectors in $MC_1(d,k,\varepsilon)$ is similar to (2.19) for $BW_{2^d}$, but is not as definitive. We can, however, obtain an analogue of the standard generating sets (2.18) with the top-closure concept.

5 Lattices with binary bases

To prove our main results about short vectors in the lattices $MC_1(d,k,\varepsilon)$, we begin with a general theory for lattices with a binary basis. Later, we shall specialize to the Barnes-Wall lattices.
Definition 5.1. Let $L$ be an integral lattice and $M$ in another lattice in $\mathbb{Q} \otimes L$ so that $L \leq \mathbb{Z}^{[1]}_{2} \otimes M$. Let $q \geq 0$ be an integer. Define $L(q) := 2^{-q}M \cap L$. Call this the $M$-level $q$ sublattice of $L$. The level of $0 \neq x \in L$ with respect to $M$ is $\min \{k \geq 0 \mid x \in L(k)\}$. The $q$-th layer of $L$ is $L(q)/L(q-1)$. If $S$ is a subset of $\mathbb{Q} \otimes L$ which is $\mathbb{Q}$-linearly independent and such that its $\mathbb{Z}^{[1]}_{2}$-span contains $L$, we call $S$ a binary basis and define level of $x \in L$ with respect to $S$ to be the level of $x \in L$ with respect to $\text{span}_{\mathbb{Z}}(S)$. We do not assume that $S$ is an orthogonal set.

Notation 5.2. If $n \in \mathbb{Z}^{[1]}_{2}$ is nonnegative, its 2-adic expansion is an expression $n = \sum_{i=p}^{q} a_{i}2^{i}$, where the $a_{i}$ come from $\{0, 1\}$. When $n \in \mathbb{Z}^{[1]}_{2}$ is negative, its 2-adic expansion is $\sum_{i=p}^{q} -a_{i}2^{i}$, where $-n = \sum_{i=p}^{q} a_{i}2^{i}$ is the 2-adic expansion of the nonnegative rational $-n$. The level of $n$ is $-\infty$ if $n = 0$ and is otherwise $-\min \{i \mid a_{i} \neq 0\}$.

Notation 5.3. Let $L$ be a lattice of rank $n$ with $S$, a linearly independent subset $v_{1}, \ldots, v_{n}$. Then $x \in L$ has a unique expression $x = \sum_{i} c_{i}v_{i}$, for rational numbers $c_{i}$. We assume that $S$ is a binary basis for $L$ (5.1). Then the $c_{i}$ are in $\mathbb{Z}^{[1]}_{2}$.

We define the 2-adic expansion of $x$ to be $\sum_{i} 2^{i}(\sum_{j} a_{i,j}v_{j})$ where the $a_{i,j}$ are the 2-adic coefficients of $c_{j}$. For $x \in L$, define level($x$) to be the least integer $m$ so that the coefficients of $\sum_{i} 2^{m}c_{i}v_{i}$ are integers. We define level(0) := $-\infty$.

For $x \neq 0$, we define top($x$) = tops($x$) to be the subsum $\sum_{j} a_{m,j}v_{j}$ of the 2-adic expansion of $x$ (it is the part of the 2-adic expansion of $x$ which represents the largest denominators, $2^{m}$). Note that the definition of top($x$) depends on the binary basis, not on the sublattice it spans. Define bot($x$) := $x - \text{top}(x)$.

Remark 5.4. The top of a vector may not be in the lattice. Consider the lattice $L$ in $\mathbb{Q}^{2}$ which is spanned over $\mathbb{Z}$ by $(1, 0), (0, 1), (\frac{1}{2}, \frac{1}{4})$. For $S$, take $\{(1, 0), (0, 1)\}$. We claim that top((\frac{1}{2}, \frac{1}{4})) = (0, \frac{1}{4})$ is not in $L$. If $(0, \frac{1}{4}) = a(1, 0) + b(0, 1) + c(\frac{1}{2}, \frac{1}{4})$, we may assume that $c \in \{0, 1, 2, 3\}$. Clearly, $c$ is $1(\text{mod } 4)$, so $c = 1$. Then the right side has first coordinate a noninteger, contradiction.

Definition 5.5. Suppose that the lattice $L$ has binary basis $S$. If for all $x \in L$, top($x$) $\in L$, then we say $L$ has top-closure.

Remark 5.6. Since $(x, x) \geq (\text{top}(x), \text{top}(x))$, if the lattice has top-closure, its minimal vectors have the form top($x$), with sign changes at some of the coordinates. This is helpful.
Proposition 5.7. Let $L := BW_{2d}$ and take a subset of a lower frame (2.20) to be our binary basis. If $v \in L, \text{top}(v) \in L$ and $\text{top}(v)$ is in the second standard generating set (2.18). Therefore the Barnes-Wall lattices have top-closure with respect to this binary basis and the tops lie in the first standard generating set.

Proof. Let $v \in L$ and suppose that $v$ has level $m \geq 1$. Then $v + L(m-1)$ is a nonzero vector in $L(m)/L(m-1)$ which is isomorphic to the code $RM(d-2m,d)$ [3]. Thus, $\text{top}(v)$ corresponds to a codeword. In fact $\text{top}(x)$ is in the first standard generating set for $L$ (2.18). □

Lemma 5.8. Suppose that the lattice $L$ is the sum of lattices $M,N$ and that there is a binary basis $S$ of $L$ which is also a binary basis for each of $M,N$. If $M, N$ have top closure, then for all $x \in L$, there exists an expression $x = y + z$, where $y \in M, z \in N$ and $y = 0$ or $z = 0$ or $\text{level}(x) = \text{level}(y) = \text{level}(z)$.

Proof. We assume that $x \neq 0$ there is no such expression with $y = 0$ or $z = 0$. Suppose that $\text{level}(x) = \text{level}(y) = \text{level}(z)$ is not true. Then $\text{level}(x) < \text{level}(y) = \text{level}(z)$ and $\text{top}(y) = \text{top}(z)$.

Since $M$ and $N$ have top closure, we may replace the expression $x = y + z$ by $x = y' + z'$, where $y' = y - \text{top}(y)$ and $z' := z + \text{top}(y)$. Now, $\text{level}(y') < \text{level}(y)$. Therefore, $\text{level}(z') < \text{level}(z)$. It follows from the form of the linear combination of $\text{top}(z'), \text{top}(y)$ and from top closure of $M$ and $N$, that $\text{top}(z) = \text{top}(y) \in M \cap N$. Consequently, $z' \in N$.

So we have reduced the levels of the two summands on the right side of $x = y + z$. We may continue this reduction until we achieve $y = 0$, $z = 0$ or the equality of the three levels after finitely many steps. □

Lemma 5.9. Suppose that $S$ is a binary basis of the lattice $L$ and that $T$ is a subset of $S$. Let $P$ be the orthogonal projection to the $\mathbb{Q}$-span of $T$.

(i) If $L$ has top closure, so does $L \cap \text{span}_\mathbb{Q}(T)$

(ii) If $L$ has top closure, so does $P(L)$ with respect to its binary basis $T$.

Proof. (i) Trivial. (ii) Let $0 \neq x \in P(L)$. There is $y \in L$ so that $x = P(y)$. We claim that we may assume $\text{level}(x) = \text{level}(y)$. Clearly, $\text{level}(x) \leq \text{level}(y)$. If the inequality is strict, we may replace $y$ by $y - \text{top}(y) \in L$ as in the proof of (5.8). Continuing, we achieve the claim. If $\text{level}(x) = \text{level}(y)$, $\text{top}(x) = P(\text{top}(y))$ and the latter is in $P(L)$. □
Lemma 5.10. We define $L := BW_{2d}$ and let $L(q)$ be the level $q$ sublattice with respect to a lower frame (2.20). In the action of $R_d$ on the section $L(q)/L(q-1) \cong 2^{2d}$, the $\varepsilon_C$ act trivially and the nonidentity translations act nontrivially.

Proof. Suppose that $A \in RM(d-2q,d)$. Then $2^{-q}v_A$ is a standard generator (2.18). Let $C \in RM(2,d)$. We calculate $2^{-q}v_A\varepsilon_C = 2^{-q}v_A - 2 \cdot 2^{-q}v_{A\cap C}$ and note that $A \cap C \in RM(d-2q,2,d)$. Since $2^{-q-1}v_{A\cap C} \in L(q-1)$, the $\varepsilon_C$ act trivially.

If we take $A$, an affine $2q$-space and a translation $\tau \neq 1$, then $A + A\tau$ is an affine $(2q+1)$-space and so is not in $RM(d-(2q+2),d)$. Therefore, $2^{-q}v_A + L(q-1) \in L(q)/L(q-1)$ is not fixed by $\tau$. □

5.1 $MC_1(d,k,\varepsilon)$ has top closure

The main result of this subsection is (5.18), which helps determine short vectors (5.6). We use the notation of (4.7).

Remark 5.11. The projection map $P^\varepsilon$ takes $v_A$ to $v_{A\cap Z}$. If we apply $P^\varepsilon$ to the equation $v_A + v_B = v_{A+B} + 2v_{A\cap B}$, we get $v_{A\cap Z} + v_{B\cap Z} = v_{(A+B)\cap Z} + 2v_{A\cap B\cap Z}$.

In case $A \subseteq Z$, this reads $v_A + v_{B\cap Z} = v_{(A+B)\cap Z} + 2v_{A\cap B} = v_A + v_{B\cap Z} = v_{A+(B\cap Z)} + 2v_{A\cap B}$.

Lemma 5.12. Let $m \geq 0$. Suppose that $2^{-m}v_A$, $2^{-m}v_B$ are in $P^\varepsilon(L)$. Then $2^{-m}v_{A+B} \in P^\varepsilon(L)$ and $2^{-m+1}v_{A\cap B} \in 2P^\varepsilon(L)$.

Proof. If $A = 0$ or $B = 0$, this is trivial. If $m = 0$, the statements are obvious since $v_i \in L^\varepsilon(t)$ for all $i \in Z$.

Suppose that $m \geq 1$. It suffices to prove that $2^{-m+1}v_{A\cap B} \in P^\varepsilon(L)$. Let $i = 2a, j = 2b \in [0,d]$ be the respective RM-levels (2.6), i.e., the smallest nonnegative even integers which satisfy $A \in RM(d-i,d), B \in RM(d-j,d)$. Then $a \geq m$ and $b \geq m$. Also, $A \cap B \in RM(d-i-j,d)$ (2.5), whence $2^{-a-b}v_{A\cap B} \in L$. We are done if $a + b \geq m + 1$. This follows since $a \geq m$, $b \geq m$ and $m \geq 1$. □

Lemma 5.13. $L^\pm(t)$ and $P^\pm(L)$ have top closure.

Proof. Use (5.9)(i, ii). □
Lemma 5.14. Recall (4.7). Suppose that \( w \in L \) and \( \text{top}(w) = 2^{-j}v_S \). Then

(i) if \( S = S_T \), \( \text{top}(w\xi) = 2^{-j+1}v_{S \cap H} \) and \( \text{top}(w)\xi = -2^{-j+1}v_{S \cap H_T} \).

(ii) if \( S \neq S_T \), \( \text{top}(w\xi) = 2^{-j}v_{S+S_T} \), \( \text{top}(w)\xi = 2^{-j}v_{S+S_T} - 2^{-j+1}v_{S \cap S_T} \).

(iii) For all \( w \in P^+(L) \), \( \text{top}(w) \xi - \text{top}(w\xi) \in L\xi \).

Proof. A hint of the proof is suggested by the matrix calculations:

\[
(1, 0) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = (1, 1), \quad (0, 1) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = (1, -1)
\]

\[
(1, 1) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = (2, 0), \quad (1, -1) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = (0, 2)
\]

We write out details using the RM-notations. Recall the concepts of BW-level and RM-level (2.6).

We may assume that \( j \geq 1 \) and that \( \frac{1}{2}w \not\in L \). Then \( S \) has BW-level at most \( j \) and so \( S \in RM(d - 2j, d) \).

Let \( \text{top}(\text{bot}(w)) = 2^{-j+r}v_D \), for some \( r \geq 1 \). By top closure, this is in \( L \), so that the BW-level of \( D \) is at most \( j - r \leq j - 1 \). We have \( w = 2^{-j}v_S + 2^{-j+r}v_D + u \), where \( \text{level}(u) \leq j - 2 \). Consequently,

\[ w\xi = 2^{-j}v_S \xi + 2^{-j+r}v_D \xi + u\xi. \quad (*) \]

(i) Since \( S = S_T \), we have

\[
\text{top}(w)\xi = 2^{-j}v_S \xi = 2^{-j+1}v_{S \cap H_T}. \quad (i.1)
\]

\[
2^{-j+r}v_D \xi = 2^{-j+r}v_{D + D_T} - 2^{-j+r+1}v_{(D \cap H) T_T}. \quad (i.2)
\]

Therefore, \( \text{top}(w\xi) = 2^{-j+1}v_{(S \cap H) T_T} \) if \( r \geq 2 \) and \( \text{top}(w\xi) = 2^{-j+1}v_{(S \cap H) T_T + D_T(\tau - 1)} \) if \( r = 1 \). In these respective cases, \( \text{top}(w\xi) - \text{top}(w)\xi = 2^{-j+1}v_S \) and 

\[
\text{top}(w\xi) - \text{top}(w)\xi = 2^{-j+1}v_{S + D_T(\tau - 1)}. \]

Since \( 2^{-j+1}v_S \in 2L \leq L\xi \), we are done if \( r \geq 2 \). It suffices to assume \( r = 1 \) and prove that \( 2^{-j+1}v_{S + D_T(\tau - 1)} \).

There exists \( T \in RM(d - 2j + 1, d) \) so that \( S = T(\tau - 1) \) (2.8). Therefore, \( T + D \in RM(d - 2j + 2) \) and \( 2^{-j+1}v_{S + D(\tau - 1)} = 2^{-j+1}v_{T + D(\tau - 1)} = 2^{-j+1}v_{T + D(\tau + 1)} - 2^{-j}v_{T \cap D} \). Since \( T \cap D \in RM(d - (2j - 1) - (2j - 2), d) = \text{RM}(d - (4j - 3)), d) \) and \( 4j - 3 = 2(j - 1) + 1, 2^{-j-1}v_{T \cap D} \in L \) and so \( 2^{-j}v_{T \cap D} \in 2L \leq L\xi \). Finally, we note that by commutator density (2.27), \( 2^{-j+1}v_{T + D(\tau + 1)} \in L\xi \). This proves (iii) in case \( S = S_T \).
(ii) This is trivial since $S + S \tau \neq 0$. In more detail, we calculate $\text{top}(w) \xi = 2^{-j}(v_{S + S \tau} - 2 \cdot v_{S \cap S \tau})$. Also, $\text{top}(w \xi) = 2^{-j}v_{S + S \tau}$. By (2.5), $2^{-j}v_{S \cap S \tau} \in L$ so that $\text{top}(w) \xi - \text{top}(w \xi) = 2^{-j+1}v_{S \cap S \tau} \in 2L \leq L \xi$, proving (iii) in case $S \neq S \tau$. □

Corollary 5.15. If $x \in P^\varepsilon(L)[1]$ has level $m \geq 1$, $\text{top}(x) = 2^{-m}v_E$, where $E \in RM(d - 2m + 1, d)$.

Proof. This follows from the possible forms of $\text{top}(w \xi)$ in (5.14). □

Lemma 5.16. $L \xi$ and $P^\varepsilon(L) \xi$ have top closure.

Proof. Let $w \in P^\varepsilon(L)$. From (5.14), (iii), we have $\text{top}(w \xi) - \text{top}(w) \xi \in L \xi$. Therefore, top closure of $L$ implies top closure for $L \xi$. The linear transformations $P^\varepsilon$ commute with $\xi$, whence the $P^\varepsilon(L) \xi$ have top closure (5.9) (ii). □

Corollary 5.17. For all $j \geq 0$, $P^\varepsilon(L)[j]$ has top closure.

Proof. If $P^\varepsilon(L)[\ell]$ has top closure, so does $P^\varepsilon(L)[\ell + 2]$ The corollary follows from (5.7), (5.9) and (5.16). □

Proposition 5.18. For all $j \geq 0$, $L^\varepsilon(t) + P^\varepsilon(L)[j]$ has top closure. In particular, $MC_1(d, k, \varepsilon)$ has top closure.

Proof. Let $M := L^\varepsilon(t), N := P^\varepsilon(L)[j]$. For all $j$, $N$ has top closure, by (5.12) and (5.16). We take $x \in M + N$ and show that $\text{top}(x) \in M + N$.

By (5.8), we may assume that $x = y + z, y \in M, z \in N$ and each of $x, y, z$ has level $m$, for some integer $m \geq 0$. The statement is trivial for $m = 0$, so we assume that $m \geq 1$.

Let $2^{-m}v_A = \text{top}(y), 2^{-m}v_B = \text{top}(z)$. Then $A + B \neq 0$ and $A \in RM(d - 2m, d)$ and $B \in RM(d - 2m + 1, d)$ (5.15). By (2.5), $A \cap B \in RM(d - (2m + (2m - 1)), d) = RM(d - (4m - 1), d) \leq RM(d - 4m + 2, d)$, whence $2^{-2m+1}v_{A \cap B} \in L$ (2.18).

We have
\[2^{-m}v_A + 2^{-m}v_B = 2^{-m}v_{A+B} - 2^{-m+1}v_{A \cap B} \quad \text{(*)}\]

Since $2^{-m+1}v_{A \cap B} = 2^m 2^{-2m+1}v_{A \cap B}$ and $m \geq 1$, this is in $2L$. Since $A \cap B \subset B \subset Z$, we have $2^{-m+1}v_{A \cap B} \in 2L \cap L^\varepsilon(t) = 2L^\varepsilon(t)$. It follows that $\text{top}(x) = 2^{-m}v_{A+B}$. By (*), $\text{top}(x) \in M + N$. □
Corollary 5.19. Suppose that $0 \neq x \in MC_1(d,k,\varepsilon)$ has level $m$. Then $\top(x) = 2^{-m}v_B \in MC_1(d,k,\varepsilon)$, where $B \in RM(d-2m+1,d)$. Furthermore, given $\tau = \tau_c$ in $0 \neq c \in \text{core}(Z)$, there is a decomposition $B = S + T$, where

(i) $S \in RM(d-2m,d), T \in RM(d-2m+1,d)$;

(ii) $S \subseteq \Omega, T \subseteq \Omega$; and

(iii) $T$ is $\tau$-invariant or $T$ has form $A \cap H$ where $A \in RM(d-2m+1,d), A \subseteq \Omega, A$ is $\tau$-invariant and $H$ is a hyperplane transverse to $\tau$ (i.e., transverse to $\{0,c\}$ in $\Omega$).

Proof. Use (5.18). □

Remark 5.20. From (5.18), we deduce that minimal vectors of $MC_1(d,k,\varepsilon)$ at level $m$ have the form $2^{-m}v_B\varepsilon c$, where $B$ is as in (5.19) and $C$ is a subset of $\Omega$. This is less precise than (4.11). The analysis in the top-closure theory for $MC_1(d,k,\varepsilon)$ will help in the classification of small norm vectors in the next section.

5.2 Equations with codewords and commutation

We collect a few results about expressions of the form $B = S + T \in RM(d - 2m, d)$ as in (5.19).

Lemma 5.21. Suppose that $B \in RM(i,d), B = S + T \in RM(d - 2m, d)$ as in (5.19). Let $r$ be a real number so that $|B| \leq 2^r$. If $d > r + i$, then $B$ is $\tau$-invariant.

Proof. We may assume that $i \geq 1$. We have $B(\tau - 1) \in RM(i - 1, d)$, which has minimum weight $2^{2i - (i - 1)}$. Since $|B(\tau - 1)| \leq 2^{r+1}$, if $B(\tau - 1) \neq 0$, then $d - i + 1 \leq r + 1$, or $d \leq r + i$, contrary to hypothesis. Therefore $B(\tau - 1) = 0$, i.e., $B$ is $\tau$-invariant. □

Corollary 5.22. Assume the hypotheses of (5.21). If $0 \neq |B| \leq 2$ and $i = d - 2$, then $B$ is $\tau$-invariant.

Proof. Take $r = 1$. □

Lemma 5.23. Suppose $\tau = \tau_c$, for $c \in \text{core}(Z)$ and $c \neq 0$. Suppose $B \in RM(d - 2m + 1, d)$ is fixed by $\tau$. Then $|B| \geq 2^{2m-1}$.

Proof. Let bars denote images in the quotient code $\Omega/\Gamma$ (2.11), where $\Gamma = \{0,c\}$. Then $\bar{B}$ is a nontrivial element of $RM(d - 2m + 1, d - 1) = RM(d - 1 - (2m - 2), d - 1)$, so has weight at least $2^{2m-2}$. This implies $|B| \geq 2^{2m-1}$. □
6 $MC_1(d, k, \varepsilon)$ short vectors, level at most 2

By (5.20), a minimal vector of $MC_1(d, k, \varepsilon)$ is a vector of the form $2^{-m}v_B \varepsilon C$, for some $m \geq 0$, some $B \in RM(d-2m+1, d)$ and some $C \subseteq \Omega$. We can say more about short vectors in the first few levels.

Recall the concept of level (5.1). Vectors of level 0 are in $BW_{2d}$, so their norms are 0 or are at least $2^\delta$. The set of level 0 norm $2^\delta$ vectors is just $\{\pm v_i | i \in \mathbb{Z}\}$.

6.1 Norms for level 1

We display a set of norm $2^\delta - 1$ vectors, which turn out to be the only level 1 vectors in $MC_1(d, k, \varepsilon)$ of norm less than $2^\delta$.

**Lemma 6.1.** Suppose that the level of $v_B \in MC_1(d, k, \varepsilon)$ is 1.

(i) $|B|$ is even.

(ii) If $(x, x) < 2^\delta$, then $B$ is a 2-set and $B$ is stabilized by some $\tau_c \neq 1$.

**Proof.**

(i) Trivial since $B \in RM(d-2m+1, d)$ and $m = 1$.

(ii) Use (5.22). □

**Lemma 6.2.** The set of level 1 vectors of $MC_1(d, k, \varepsilon)$ of norm less than $2^\delta$ consists of all $\pm \frac{1}{2}v_i + \frac{1}{2}v_{i+c}$, for $c \neq 1, c \in core(\mathbb{Z})$ and $i \in \Omega$. These have norm $2^\delta - 1$.

**Proof.** We get a list of candidates from (6.1)(ii). We need to see that all the vectors of indicated form are actually in $MC_1(d, k, \varepsilon)$. By (2.3), there exists $E \in RM(d-2, d)$ so that $F := E \cap \mathbb{Z}$ is an odd set. Therefore $F(\tau - 1)$ has cardinality $2(mod 4)$. By (2.9)(ii), there exists $S \in RM(d-2, d)$ so that $B = S + F(\tau - 1)$ is a 2-set, and such a 2-set is $\tau$-invariant (5.22) and so is one of the indicated $i + c$.

**Remark 6.3.** We recall an elementary result about positive definite integral lattices. Let $J$ be such a lattice. Call $x \in J, x \neq 0$ decomposable if there exist nonzero $y, z \in J$ so that $x = y + z$. If $X$ is the set of indecomposable vectors, we define a graph structure by connecting two members of $X$ with an edge if they are not orthogonal. We therefore get $X$ as the disjoint union of connected components $X_i$. If $J_i$ is the sublattice spanned by $X_i$, then $X$ is their orthogonal direct sum. If $Y$ is any orthogonal direct summand of $J$, $Y$ is a sum of a subset of the $J_i$. 28
Corollary 6.4. The vectors of (6.2) span a sublattice which is an orthogonal direct sum of scaled $D_{2d-2k}$ root lattices. This sublattice has finite index in $MC_1(d, k, \varepsilon)$.

**Proof.** Consider the natural graph on this set of vectors where edges between distinct vectors are based on nonorthogonality. The connected components span lattices of type $D$. □

6.2 Norms for level 2

For the moment, $d \geq 5$ is odd and arbitrary.

We shall prove that the minimum vectors at this level have the form $2^{-m}v_B \in \Omega$, where $B$ is an affine 3-space and $C \subseteq \Omega$. Furthermore, the set of $B$ which occur this way are all translates of $core(Z)$ and partition $\Omega$.

We can discuss the situation with level 2 minimal vectors completely in the case of $d$ odd.

Proposition 6.5. Suppose that $d \geq 5$ and $d - 2k \geq 3$. If the norm of the level 2 vector $x \in MC_1(d, k, \varepsilon)$ is $2^{\delta - 1}$, then $\top(x) = \frac{1}{4}v_B$, where $B$ is an affine 3-space. Furthermore, such vectors occur in level 2.

**Proof.** It suffices to prove that such vectors occur. Take a translate $K$ of $core(Z)$ so that $K \subseteq Z$. Let $T$ be an affine 2-space in $K \cap H$. The lattice vector $u := \frac{1}{2}v_T$ is in $L^\varepsilon(t)$ and so $u(f-1)^{-1} \in MC_1(t, f, \varepsilon)$. Since $(f-1)^2 = -2f$, $(f-1)^{-2} = -\frac{1}{2}f^{-1} = \frac{1}{2}f$ and $(f-1)^{-1} = \frac{1}{2}f(f-1)$. Therefore $u(f-1)^{-1} = \frac{1}{2}u(f-1)f = -\frac{1}{2}(v_T + v_{T'})f = -\frac{1}{4}v_{T+T'} + \frac{1}{2}v_{T'}$. By top closure (5.18), $\frac{1}{4}v_{T+T'} \in MC_1(t, f, \varepsilon)$. □

7 Decomposability and indecomposability

We prove that the first cousins are orthogonally decomposable for $k = 1$ and indecomposable for $k \geq 2$. As in (4.7), $t$ has positive trace.

Proposition 7.1. Let $k = 1$. The lattice $MC_1(d, 1, -)$ is isometric to $BW_{2d-1}$.

**Proof.** By ancestral theory [13], $L^-(t) \cong BW_{2d-2}[1]$. By (3.18)(iii), $MC_1(d, 1, -) \cong 2^{-\frac{1}{2}}L^-(t) \cong BW_{2d-1}$. □
Proposition 7.2. Let \( k = 1 \). The lattice \( MC_1(d, 1, +) \) is isometric to \( BW_{2d-1} \perp BW_{2d-1} \perp BW_{2d-1} \).

Proof. By hypothesis, \( k = 1 \). Thus, \( Z' \) is the complement in \( \Omega \) of a codimension 2 affine space. There are three affine hyperplanes contained in \( Z' \). Call them \( Z_1, Z_2, Z_3 \) and let \( Z_{ij} \) denote the intersection of \( Z_i \) and \( Z_j \).

The proof is a consequence of the theory of [13, 14]. For a subset \( T \) of \( \Omega \), we let \( L(T) \) be the set of vectors in \( L \) whose support is contained in \( T \). Then \( L(Z_i) \) is a scaled \( BW_{2d-1} \). The sublattice \( L(Z) \) has codimension \( 2^{d-2} \) in the orthogonal direct sum \( \frac{1}{2} L(Z_{12}) \perp \frac{1}{2} L(Z_{23}) \perp L(Z_{31}) \). Furthermore, a set of coset representatives for \( L(Z_i) \perp L(\Omega + Z_i) \) in \( L \) is just the set \( S \) of all \( x + xu \), where \( u \) is a suitable involution interchanging \( L(Z_i) \) and \( L(\Omega + Z_i) \) and where \( x \in L(Z_i)[-1] \).

It follows that the set \( P^e(S)(f - 1) \) represents all the cosets of \( L(Z) \) in \( \frac{1}{2} L(Z_{12}) \perp \frac{1}{2} L(Z_{23}) \perp L(Z_{31}) \). \( \square \)

Lemma 7.3. Suppose that \( M \) is an integral lattice and \( N \) a finite index sublattice. Suppose that \( N \) is spanned by vectors which are indecomposable in \( M \) and that \( N \) is orthogonally indecomposable. Then \( M \) is orthogonally indecomposable.

Proof. The hypotheses on \( M \) and \( N \) imply that \( N \) meets every indecomposable summand of \( M \) nontrivially. See (6.3). \( \square \)

Lemma 7.4. Suppose that \( H \) is a hyperplane which is transverse to core \((Z)\). Set \( v := 2^{-\delta}v_H \), a minimal vector in \( L = BW_{2d} \). Then \( P^e(v) \) has norm \( 2^{d/2} = r2^\delta \), for some \( r \in [\frac{1}{1}, \frac{3}{1}] \). Also, \( P^e(v)(f - 1) \) has norm \( r2^{\delta+1} = s2^{\delta-1} \), for some \( s \in [1, 3] \). Therefore, if we write \( P^e(v)(f - 1) = w_1 + \cdots + w_n \) as an orthogonal sum of indecomposable nonzero vectors, \( n \leq 3 \).

Proof. Use the formula for \( |Z'| \) (4.7), (4.9) and the fact that \( P^e(v)(f - 1) \in MC_1(d, k, \varepsilon) \). \( \square \)

Proposition 7.5. Suppose that \( d \geq 7 \) is odd and \( k \geq 2 \).

(i) The level 1 minimal vectors are indecomposable in \( MC_1(d, k, \varepsilon) \). The sublattice of \( MC_1(d, k, \varepsilon) \) which they span is an orthogonal direct sum of scaled type \( D_{2d-2k} \)-lattices.

(ii) When \( d \geq 7 \) and \( d - 2k \geq 5 \), the lattice spanned by the level 2 minimal vectors (which have norms \( 2^{\delta-1} \)) is orthogonally indecomposable and has finite index in \( MC_1(d, k, \varepsilon) \). Therefore, \( MC_1(d, 1, \varepsilon) \) is orthogonally indecomposable.
Proof. (i) The first statement is trivial since they are minimal vectors in $MC_1(d, k, \varepsilon)$. The second statement follows from analysis as in the proof of (6.5).

(ii) Let $L_1, \ldots, L_r$ be the set of scaled type $D_{2d-2k}$-lattices as described in (i). Each is orthogonally indecomposable since $d - 2k \geq 3$.

Take a vector hyperplane $H$ and vector $v$ as in (7.4). Then $v$ has nonzero inner product with vectors of each $L_i$ and so does $P_\varepsilon(v)(f - 1)$. If we write $P_\varepsilon(v)(f - 1) = w_1 + \cdots + w_n$, then $n \leq 3$. For each $i$, there exists $j$ so that $L_i$ has nonzero inner products with $w_j$. Since $r = 2^{2k-1} + \varepsilon 2^{k-1}$ and $k \geq 2$, $r \geq 6$. Therefore, there exists a pair of distinct indices $i, i'$ and an index $j$ so that both $(L_i, w_j)$ and $(L_{i'}, w_j)$ are nonzero. Therefore in the graph of indecomposable vectors (6.3), the minimal vectors of $L_i$ and $L_{i'}$ are in the same component. Now we quote double transitivity of $Sp(2k, 2)$ on the set of $L_i$ [12] to deduce that all minimal vectors of $L_1 \perp L_2 \perp \cdots \perp L_r$ are in the same component. This proves that $MC_1(d, k, \varepsilon)$ is indecomposable. □

8 More distant cousins

The author has considered variations of the formula for first cousins. Many interesting high dimensional, unimodular lattices with moderately high minimum norms may be created in the midwest style. Precise analysis of their properties would be challenging, however.

One variation creates an even unimodular rank 24 overlattice of $L^+(t)$ for $L \cong BW_{24}$ and $tr(t) = 8$. That overlattice has minimum norm 4, so is isometric to the Leech lattice.

Here is a sketch of the construction. In $L^+(t)$, there is a sublattice $M = M_1 \perp M_2 \perp M_3$, where $M_i \cong \sqrt{2} E_8$, for $i = 1, 2, 3$. Let $f$ be a lower fourvolution on $L$ which commutes with $t$ and fixes each $M_i$. Then $L^+(t)(f - 1) \leq M$ and $P^+(L)(f - 1) \leq L^+(t)$. Let $\gamma$ be an isometry of $M$ which stabilizes each $M_i$ and satisfies $M_i(f - 1) \cap M_i(f - 1) \gamma = 2M_i$ and (consequently) that $M_i(f - 1) + M_i(f - 1) \gamma = M_i$ (see (2.21) and the ancestral theory [13]). Then $L^+(t) + P^+(L)(f - 1) \gamma$ is isometric to the Leech lattice. There is similarity in spirit to [16, 21].

It is well-known that the Leech lattice contains sublattices isometric to $BW_{24}$ [4], [10]. The above result links the Leech lattice and $BW_{25}$. 

31
References


