The rank two lattice type vertex operator algebras $V_L^+$ and their automorphism groups

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Abstract. Let $L$ be a positive definite even lattice and $V_L^+$ be the fixed points of the lattice VOA $V_L$ associated to $L$ under an automorphism of $V_L$ lifting the $-1$ isometry of $L$. For any positive rank, the full automorphism group of $V_L^+$ is determined if $L$ does not have vectors of norms 2 or 4. For any $L$ of rank 2, a set of generators and the full automorphism group of $V_L^+$ are determined.

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\end{align*}\]
1 Introduction

This article continues a program to study automorphism groups of vertex operator algebras. See references in the survey [G2] and the more recent articles [G1], [DG1], [DG2], [DGR] and [DN1].

Here we investigate the fixed point subVOA of a lattice type VOA with respect to a group of order 2 lifting the $-1$ map on a positive definite lattice. We can obtain a definitive answer for the automorphism group of this subVOA in two extreme cases. The first is where the lattice has no vectors of norms 2 or 4, and the second is where the lattice has rank 2.

We use the standard notation $V_L$ for a lattice VOA, based on the positive definite even integral lattice, $L$. For a subgroup $G$ of $Aut(L)$, $V_L^G$ denotes the subVOA of points fixed by $G$. When $G$ is a group of order 2 lifting $-1_L$, it is customary to write $V_L^+$ for the fixed points (though, strictly speaking, $G$ is defined only up to conjugacy; see the discussion in [DGH] or [GH]).

The rank 2 case is a natural extension of work on the rank 1 case, where $Aut(V_L^G)$ was determined for all rank 1 lattices $L$ and all choices of finite group $G \leq Aut(V_L)$. The styles of proofs are different. In the rank 1 case, there was heavy analysis of the representation theory of the principal Virasoro subVOA on the ambient VOA. In the rank 2 case, there is a lot of work on idempotents, solving nonlinear equations as well as work with several subVOAs associated to Virasoro elements. For rank 2, the case of nontrivial degree 1 part is harder to settle than in rank 1.

Our strategy follows this model. Let $V$ be one of our $V_L^+$. We get information about $G := Aut(V)$ by its action on the finite dimensional algebra $A := (V_2, 1^*)$. We take a subset $S$ of $A$ which is $G$-invariant and understand $S$ well enough to limit the possibilities for $G$ (usually, there are no automorphisms besides the ones naturally inherited from $V_L$). A natural choice for $S$ is the set of idempotents or conformal vectors. Usually, $S$ spans $A$, or at least generates $A$. In the main case of rank 2 lattice, we prove that $Aut(V)$ fixes a subalgebra of $A$ which is the natural $M(1)_2^+$.

The structure of $V$ is controlled by $M(1)_2^+$, which is generated by $M(1)_2^+$ and its eigenspaces, so we eventually determine $G$.

For several results, we give more than one proof.

For the case of a lattice $L$ without roots, the automorphism group of $V_L^+$ was studied in the recent article [S].

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# Background Definitions and Notations

**Notation 2.1.** Let $L$ be an even integral lattice. For an integer $m$, define $L_m := \{ x \in L | (x, x) = 2m \}$. Let $H := \mathbb{C} \otimes L$, the ambient complex vector space. For a subset $S$ of $L$, define $\text{rank}(S)$ to be the rank of the sublattice spanned by $S$.

**Definition 2.2.** For a lattice, $L$, the group of automorphisms of the free abelian group $L$ which preserves the bilinear form is called the group of automorphisms, the isometry group, the group of units or the orthogonal group of $L$. This group is denoted $\text{Aut}(L)$ or $O(L)$. We will use the notation $O(L)$ in this article, as well as the associated $SO(L)$ for the elements of determinant 1, $PO(L)$ for $O(L)/\{\pm 1\}$ and $PSO(L)$ for $SO(L)/SO(L) \cap \{\pm 1\}$.

**Definition 2.3.** For an even integral lattice, $L$, we let $\hat{L}$ be the 2-fold cover of $L$ described in [FLM], [DGH], [GH]. We may write bars for the map $\hat{L} \to L$. The group of automorphisms, the isometry group, the group of units or orthogonal group is the set of group automorphisms of $\hat{L}$ which preserve the bilinear form on the quotient of $\hat{L}$ by the normal subgroup of order 2. It is denoted $\text{Aut}(\hat{L})$ or $O(\hat{L})$ and has shape $2^{\text{rank}(L)}O(L)$. We use bars to denote the natural map $O(\hat{L}) \to O(L)$.

We list some notations for work with lattice type VOAs.

**List of Notations**

- $D(L)$: the discriminant group of the integral lattice $L$ is $D(L) := L^*/L$.
- $e^\alpha$: standard basis element for $\mathbb{C}[L]$.
- $\text{FVOA}$: framed vertex operator algebra [DGH].
- $\text{LVOA}$: lattice vertex operator algebra [FLM].
- $\text{LVOA type}$: the fixed points of a lattice vertex operator algebra under a finite group of automorphisms [DG1, DGR].
- $\text{LVOA}^+$: $V_L^+$ for an even lattice $L$.
- $\text{LVOAG}(L)$: the subgroup of $Aut(V_L)$, for an even integral lattice $L$, as described in [DN1]; it is denoted $\mathbb{N}(\hat{L})$ and is an extension of the form $T.Aut(L)$ (possibly nonsplit), where $T$ is a natural copy of the torus $\mathbb{C} \otimes L/L^*$ obtained by exponentiating the maps $2\pi x_0$, for $x \in V_1$; the quotient of this group by the normal subgroup $T$ is naturally isomorphic to $Aut(L)$. Also, $\mathbb{N}(\hat{L})$ is the product of subgroups $TS$, where $S \cong O(\hat{L})$ and $S \cap T = \{ x \in T | x^2 = 1 \} \cong \mathbb{Z}^{\text{rank}(L)}_2$. We may take $S$ to be the
centralizer in $L\text{VOAG}(L)$ of a lift of $-1$; it has the form $2^{\text{rank}(L)}.\text{Aut}(L)$ and in fact any such $S$ has this form.

Denote the groups $S, T$ by $\mathbb{O}(\hat{L})$ and $\mathbb{T}(\hat{L})$, respectively.

LVOA group for $L$ this means $L\text{VOAG}(L)$.
LVOAG this means $L\text{VOAG}(L)$, for some $L$
LVOAG$^+(L)$ this is the centralizer in $L\text{VOAG}(L)$ of a lift of $-1$ modulo the group of order 2 generated by the lift; it has the form $2^{\text{rank}(L)}.[\text{Aut}(L)/\langle-1\rangle]$; it is the inherited group
LVOAG$^+$ this means $L\text{VOAG}^+(L)$, for some $L$.
LVOA$^+$-group same as LVOAG$^+$
$M(1), M(1)^+$ See Section 3.
$\mathbb{N}(\hat{L})$ See LVOAG($\hat{L}$)
o linear map from $V$ to $\text{End}(V)$
$\mathbb{O}(\hat{L})$ See LVOAG($\hat{L}$)
$\mathbb{T}(\hat{L})$ See LVOAG($\hat{L}$)
v$_{\alpha}$ $e^{\alpha} + e^{-\alpha}$
$X$ or $X(L)$: given an even integral lattice, $L$, this is a group of shape $2^{1+\text{rank}(L)}$ for which commutation corresponds to inner products modulo 2; see an appendix of [GH].
$X\mathbb{O}$ or $X\mathbb{O}(\hat{L})$ an extension of $X$ upwards by $O(L)$.
$X\mathbb{P}O$ or $X\mathbb{P}O(\hat{L})$ a quotient of $X\mathbb{O}$ by a central involution which corresponds to $-1_L$ under the natural epimorphism to $O(L)$.

Remark 2.4. If $(L, L) \subset 2\mathbb{Z}, \hat{L} \cong L \times \langle \pm 1 \rangle$. Thus $O(\hat{L})$ contains a copy of $O(L)$ which complements the normal subgroup of order $2^{\text{rank}(L)}$ consisting of automorphisms which are trivial on the quotient group $L$ of $\hat{L}$. This splitting passes to the groups $PO(\hat{L})$ and $XPO(L)$.

3 Automorphism group of $V^+_L$ with $L_1 = L_2 = \emptyset$

In this section, we determine the automorphism group of $V^+_L$ with $L_1 = L_2 = \emptyset$ and assume only that $\text{rank}(L) > 1$. The automorphism group of $V^+_L$ in the case $\text{rank}(L) = 1$ is determined in [DG1] without any restriction on $L$. The assumption that $L_1 = L_2 = \emptyset$ ensures that any automorphism of $V^+_L$ preserves the subspace $M(1)^+_2$, which can be identified with the Jordan algebra $S^2H$.

Since $M(1)^+_2$ is generated by $M(1)^+_2$ if $\dim H > 1$ and $V^+_L$ is a direct sum of eigenspaces for $M(1)^+_2$ (cf. [AD]), the structure of $\text{Aut}(V^+_L)$ can be determined easily. We shall use a classic result.
Proposition 3.1. The automorphism group of the Jordan algebra of symmetric $n \times n$ matrices is $PO(n, \mathbb{C})$, acting by conjugation.

Proof. [J]. □

3.1 $\text{Aut}(M(1)^{\dagger})$

We first recall the construction of $M(1)^{\dagger}$. Let $H$ be a $n$-dimensional complex vector space with a nondegenerate symmetric bilinear form $(\cdot, \cdot)$ and $\hat{H} = H \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ the corresponding affine Lie algebra. Consider the induced $\hat{H}$-module

$$M(1) = U(\hat{H}) \otimes_{U(H \otimes \mathbb{C}[t] \oplus \mathbb{C}c)} \mathbb{C} \simeq S(H \otimes t^{-1} \mathbb{C}[t^{-1}]) \quad \text{(linearly)}$$

where $H \otimes \mathbb{C}[t]$ acts trivially on $\mathbb{C}$, and $c$ acts as 1. For $\alpha \in H$ and $n \in \mathbb{Z}$ we set $\alpha(n) := \alpha \otimes t^n$. Let $\tau$ be the automorphism of $M(1)$ such that

$$\tau(\alpha_1(-n_1) \cdots \alpha_k(-n_k)) = (-1)^k \alpha_1(-n_1) \cdots \alpha_k(-n_k)$$

for $\alpha_i \in H$ and $n_1 \geq \cdots \geq n_k \geq 1$. Then $M(1)^{\dagger}$ is the fixed point subspace of $\tau$.

Proposition 3.2. The automorphism group of $M(1)^{\dagger}$ is $PO(n, \mathbb{C})$.

Proof. We first deal with the case that $\dim H > 1$. Then $M(1)^{\dagger}$ is generated by $M(1)^{\dagger}_2$ (cf. [DN2]), which is a Jordan algebra under $u \cdot v = u_1v$ for $u, v \in M(1)^{\dagger}_2$. So any automorphism of $M(1)^{\dagger}$ restricts to an automorphism of the Jordan algebra $M(1)^{\dagger}_2$. On the other hand, the automorphism group of $M(1)$ is $O(n, \mathbb{C})$ [DM2], which preserves $M(1)^{\dagger}$. Clearly, the kernel of the action of $O(n, \mathbb{C})$ on $M(1)^{\dagger}$ is $\{\pm 1\}$. As a result $PO(n, \mathbb{C})$ is a subgroup of the automorphism group of $M(1)^{\dagger}$. By Proposition 3.1, any automorphism of $M(1)^{\dagger}_2$ extends to an automorphism of $M(1)^{\dagger}$.

We now assume that $\dim H = 1$. Then $M(1)^{\dagger}$ is not generated by $M(1)^{\dagger}_2$. By Lemma 2.6 and Theorem 2.7 of [DG1] for any nonnegative even integer $n$ there is a unique lowest weight vector $u^n$ (up to scalar multiple) of weight $n^2$ and $M(1)^{\dagger}$ is generated by the Virasoro vector and $u^n$. Using the fusion rule given in Lemma 2.6 of [DG1] we immediately see that the automorphism group of $M(1)^{\dagger}$ in this case is trivial. Clearly, $PO(1, \mathbb{C}) = 1$. This finishes the proof. □

3.2 $\text{Aut}(V_L^{\dagger})$

First we review from [B] and [FLM] the construction of lattice vertex operator algebra $V_L$ for any positive definite even lattice $L$. Let $H = \mathbb{C} \otimes_Z L$. Recall that $\hat{L}$ is the
canonical central extension of \( L \) by the cyclic group \( \langle \pm 1 \rangle \) such that the commutator map is given by \( c(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle} \). We fix a bimultiplicative 2-cocycle \( \epsilon : L \times L \to \langle \pm 1 \rangle \) such that \( \epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = c(\alpha, \beta) \) for \( \alpha, \beta \in L \). Form the induced \( \hat{L} \)-module

\[
\mathbb{C}\{L\} = \mathbb{C}[\hat{L}] \otimes_{\mathbb{C}[\langle \pm 1 \rangle]} \mathbb{C} \simeq \mathbb{C}[L] \text{ (linearly)},
\]

where \( \mathbb{C}[\cdot] \) denotes the group algebra and \( -1 \) acts on \( \mathbb{C} \) as multiplication by \( -1 \). For \( a \in \hat{L} \), write \( \iota(a) \) for \( a \otimes 1 \) in \( \mathbb{C}\{L\} \). Then the action of \( \hat{L} \) on \( \mathbb{C}\{L\} \) is given by:

\[
a \cdot \iota(b) = \iota(ab) \text{ for } a, b \in \hat{L}.
\]

If \( (L, \mathcal{L}) \subset 2\mathbb{Z} \) then \( \mathbb{C}\{L\} \) and \( \mathbb{C}[L] \) are isomorphic algebras. The lattice vertex operator algebra \( V_L \) is defined to be \( M(1) \otimes \mathbb{C}\{L\} \), as a vector space.

Then \( O(\hat{L}) \) is a naturally defined subgroup of \( \text{Aut}(\hat{L}) \) and \( \text{Hom}(L, \mathbb{Z}/2\mathbb{Z}) \) may be identified with a subgroup of \( O(\hat{L}) \) (see [FLM], [DN1], [GH]) and there is an exact sequence

\[
1 \to \text{Hom}(L, \mathbb{Z}/2\mathbb{Z}) \to O(\hat{L}) \to O(L) \to 1.
\]

It is proved in [DN1] that \( \text{Aut}(V_L) \) has shape \( N \cdot O(\hat{L}) \) where \( N \) is the normal subgroup of \( \text{Aut}(V_L) \) generated by \( e^{u_0} \) for \( u \in (V_L)_1 \). Note that \( \text{Hom}(L, \mathbb{Z}/2\mathbb{Z}) \) can furthermore be identified with the intersection of \( N \) and \( O(\hat{L}) \). See the List of Notations.

Let \( e : L \to \hat{L} \) be a section associated to the 2-cocycle \( \epsilon \), written \( \alpha \mapsto e_\alpha \). Let \( \theta \) be the automorphism of \( \hat{L} \) of order 2 such that \( \theta e_\alpha = e_{-\alpha} \) for \( \alpha \in L \). Then \( \theta \) extends to an automorphism of \( V_L \), still denoted by \( \theta \), such that \( \theta\vert_{M(1)} \) is identified with \( \tau \) and \( \theta \iota(a) = \iota(\theta a) \) for all \( a \in \hat{L} \). Set \( e^\alpha = \iota(e_\alpha) \). Then \( \theta e^\alpha = e^{-\alpha} \).

Let \( V_L^+ \) be the fixed points of \( \theta \). In order to determine the automorphism group of \( V_L^+ \), it is important to understand which automorphism of \( V_L \) restricts to an automorphism of \( V_L^+ \). Clearly, the centralizer of \( \theta \) in \( \text{Aut}(V_L) \) acts on \( V_L^+ \). So we get an action of \( O(\hat{L})/\langle \pm 1 \rangle \) on \( V_L^+ \). Let \( h \in H \). Then \( e^{2\pi i\theta(0)} \) preserves \( V_L^+ \) if and only if \( (h, \alpha) \equiv (h, -\alpha) \) modulo \( \mathbb{Z} \) for any \( \alpha \in L \). That is, \( h \in \frac{1}{2}L^* \) where \( L^* \) is the dual lattice of \( L \).

**Lemma 3.3.** The subgroup of \( \text{Aut}(V_L^+) \) which preserves \( M(1)_2^+ \) is just the \( \text{LVOA}^+ \)-group.

**Proof.** Let \( n := \dim(H) \). Let \( \sigma \in \text{Aut}(V_L^+) \) such that \( \sigma M(1)_2^+ \subset M(1)_2^+ \). Then \( \sigma\vert_{M(1)_2^+} \in \text{PO}(n, \mathbb{C}) \) as in 3.2. Note that \( M(1)^+ \) is generated by \( M(1)_2^+ \) as \( \text{rank}(L) > 1 \) (see the proof of Proposition 3.2). So, \( \sigma \) preserves \( M(1)^+ \).

For any \( \alpha \in L \), let \( V_L^+(\alpha) \) be the \( M(1)^+ \)-submodule generated by \( v_\alpha := e^\alpha + e^{-\alpha} \). Then \( V_L^+(\alpha) \) is an irreducible \( M(1)^+ \)-module, \( V_L^+(\alpha) \) and \( V_L^+(\beta) \) are isomorphic \( M(1)^+ \)-modules if and only if \( \alpha = \pm \beta \) (cf. [AD]). Moreover, if \( \alpha \neq 0 \) then \( V_L^+(\alpha) \) is isomorphic to \( M(1) \otimes e^\alpha \) (cf. [AD]).
Note that $V_L^+ = \sum_{\alpha \in L} V_L^+(\alpha)$. Let $S$ be a subset of $L$ such that $|S|\cap\{\pm\alpha\}| = 1$ for any $\alpha \in L$. Then for any two different $\alpha, \beta \in S$, $V_L^+(\alpha)$ and $V_L^+(\beta)$ are nonisomorphic $M(1)^+$-modules and

$$V_L^+ = \oplus_{\alpha \in S} V_L^+(\alpha)$$

is a direct sum of nonisomorphic irreducible $M(1)^+$-modules.

Let $\alpha \in L$. Since $\sigma$ preserves $M(1)^+$, it sends $V_L^+(\alpha)$ to $V_L^+(\beta)$ for some $\beta \in L$. The vector $v_\alpha$ is the unique lowest weight vector (up to a scalar) of $V_L^+(\alpha)$. This implies that $\sigma(v_\alpha) = \lambda v_\beta$ for some nonzero scalar $\lambda \in \mathbb{C}$ (depending on $\alpha$ and $\beta$).

For a vertex operator algebra $V$ and a homogeneous $v \in V$ we set $o(v) = v_{wt v-1}$ and extend to all of $V$ linearly. Note that $v_\alpha$ is an eigenvector for $o(v)$ for $v \in M(1)_+$. In fact, $o(h_1(-1)h_2(-1))v_\alpha = (h_1, \alpha)(h_2, \alpha)v_\alpha$ for $h_i \in H$. Recall the proof of Proposition 3.2. We can regard the restriction of $\sigma$ to $(V_L^+)_2 \cong M(1)_2^+$ as an element of $O(n, \mathbb{C})$, well-defined modulo $\pm 1$. Then $\sigma(h_1(-1)h_2(-1)) = (\sigma h_1)(-1)(\sigma h_2)(-1)$.

Note that $\sigma^{-1}$ is the adjoint of $\sigma$. Then,

$$(h_1, \alpha)(h_2, \alpha)\lambda v_\beta = \sigma((h_1, \alpha)(h_2, \alpha)v_\alpha) = \sigma(o(h_1(-1)h_2(-1))v_\alpha)$$

$$= o(\sigma(h_1(-1)h_2(-1)))\lambda v_\beta = (\sigma h_1, \beta)(\sigma h_2, \beta)\lambda v_\beta.$$

Since the $h_i$ are arbitrary, $\sigma \alpha = \pm \beta$. Thus $\sigma$ maps $L$ onto $L$ so induces an isometry of $L$ which is well defined modulo $(\pm 1)$.

Multiplying $\sigma$ by an element from LVOAG$^+(L)$ (which comes from $\mathbb{N}(\hat{L})$), we can assume that $\sigma|_{M(1)^+} = id_{M(1)^+}$. Then $\sigma v_\alpha = \lambda v_\alpha$ for some nonzero $\lambda_\alpha \in \mathbb{C}$. Since $V_L^+(\alpha)$ is an irreducible $M(1)^+$-module we see that $\sigma$ acts as the scalar $\lambda_\alpha$ on $V_L^+(\alpha)$. Clearly, $\lambda_\alpha = \lambda_{-\alpha}$. Note that

$$Y(v_\alpha, z)v_\beta = E^-(\alpha, z)e^{(\alpha, \beta)}e^{(\alpha, -\beta)}z^{-(\alpha, \beta)}$$

$$+ E^-(\alpha, z)e^{(\alpha, -\beta)}e^{(\alpha, \beta)}z^{-(\alpha, \beta)}$$

where

$$E^-(\alpha, z) = \exp\left(\sum_{n<0} \frac{\alpha(n)z^{-n}}{n}\right).$$

Thus, if $n$ is sufficiently negative, $(v_\alpha)_n(v_\beta) = u + v$ for some nonzero $u \in V_L^+(\alpha + \beta)$ and $v \in V_L^+(\alpha + \beta)$. This gives $\lambda_\alpha \lambda_\beta = \lambda_{\alpha + \beta} = \lambda_{\alpha - \beta}$ by applying $\sigma$ to $(v_\alpha)_n(v_\beta) = u + v$. So $\alpha \mapsto \lambda_\alpha$ defines a character of abelian group $L/2L$ of order $2^\theta$. Clearly, any character $\lambda : L/2L \to \langle \pm 1 \rangle$ defines an automorphism $\sigma$ which acts on $V_L^+(\alpha)$ as $\lambda_\alpha$.

As a result, the subgroup of $Aut(V_L^+)$ which acts trivially on $M(1)^+$ is isomorphic the dual group of $L/2L$ and is exactly the subgroup of $O(\hat{L})/\langle \pm 1 \rangle$ which we identified.
as \( \text{Hom}(L, \mathbb{Z}/2\mathbb{Z}) \). As a result, the subgroup of \( \text{Aut}(V_L^+) \) which preserves \( M(1)_2^+ \) is exactly the group \( O(\hat{L})/\langle \pm 1 \rangle \), as desired. □

**Proposition 3.4.** Let \( L \) be a positive definite even lattice such that \( L_1 = L_2 = \emptyset \). Then \( \text{Aut}(V_L^+) \) is the inherited group, i.e., the \( \text{LVOA}^+ \)-group.

**Proof.** In this case we have \( (V_L^+)_2 = M(1)_2^+ \). Thus, any automorphism of \( V_L^+ \) preserves \( M(1)_2^+ \). By Lemma 3.3, \( \text{Aut}(V_L^+) \) is the \( \text{LVOA}^+ \)-group. □

### 4 Rank 2 lattices

All lattices in this article are positive definite. Throughout this article, \( L \) denotes an even integral lattice. We recall a general result.

**Lemma 4.1.** Let \( L \) be a lattice and \( M \) a sublattice.

(i) If \( |L : M| \) is finite, \( \det(M) = \det(L)|L : M|^2 \).

(ii) If \( M \) is a direct summand of \( L \), \( L/[M + \text{ann}_L(M)] \) embeds in \( \mathcal{D}(M) \).

**Proof.** These are standard results. For example, see [G3]. □

We need to sort out rank 2 lattices by whether they contain roots or elements of order 4, due to their contributions to low degree terms of the lattice VOA. We shall use the notations 2.1.

**Lemma 4.2.** Suppose that \( \text{rank}(L_1) = 2 \). Then \( L_1 \) spans \( L \) and \( L \) is one of \( L_{A_1^2} \) or \( L_{A_2} \).

**Proof.** The span of \( L_1 \) is isometric to \( L_{A_1^2} \) or \( L_{A_2} \). Each of these is a maximal even integral lattice under containment. □

**Lemma 4.3.** Suppose that \( \text{rank}(L_1) = 1 \). Let \( r \in L_1 \) and let \( s \) generate \( \text{ann}_L(r) \). Then \( (s, s) \geq 4 \) and if \( L > \text{span}\{r, s\} \), then \( 14 \leq (s, s) \in 6 + 8\mathbb{Z} \).

**Proof.** Note that \( \mathbb{Z}r \) is a direct summand of \( L \). We have \((s, s) \geq 4 \). In case \( L > N := \text{span}\{r, s\} \), \( L/N \) has order 2, by 4.1. If \( x \) represents the nontrivial coset, \( (x, x) \geq 4 \) then \( 2(x, 2x) \geq 16 \). Also, \( (2x, 2x) \in 8\mathbb{Z} \). Since \( (x, r) \) is odd, if we write \( 2x = pr + qs \), for integers \( p, q \in \mathbb{Z} \), then \( p \) is odd and so \( q^2(s, s) \in 6 + 8\mathbb{Z} \). It follows that \( q \) is odd and \((s, s) \in 6 + 8\mathbb{Z} \). □

**Lemma 4.4.** Suppose that \( L_1 = \emptyset \) and \( \text{rank}(L_2) = 2 \). If \( r, s \) are linearly independent norm 4 elements, then they span \( L \) and have Gram matrix \( G = \begin{pmatrix} 4 & b \\ b & 4 \end{pmatrix} \), for some \( b \in \{0, \pm 1, \pm 2\} \).
Proof. If \( L \neq N := \text{span}\{r, s\} \), then \( \text{det}(N) = 16 - b^2 \) is divisible by a perfect square, whence \( b = 0 \) or \( b = \pm 2 \) and the index is 2. Actually, \( b = 0 \) does not occur here since \( \frac{1}{3}r, \frac{1}{3}s \not\in L \) implies that \( \frac{1}{3}(r+s) \in L_1 \), a contradiction. So, \( b = \pm 2 \). Clearly, \( \text{span}\{r, s\} \cong \sqrt{2}L_{A_2} \). However, any integral lattice containing the latter with index 2 is odd, a contradiction. Therefore, \( L = N \) and the Gram matrix is as above. Positive definiteness implies that \( |b| < 4 \) and rootlessness implies that \( b \neq \pm 3 \). \( \square \)

Lemma 4.5. Suppose that \( L_1 = \emptyset \) and \( \text{rank}(L_2) = 1 \). Let \( r \in L_2 \) and let \( s \) generate \( \text{ann}_L(x) \). Then \( (s, s) \geq 6 \) and \( L/\text{span}\{r, s\} \) is a subgroup of \( \mathbb{Z}_4 \).

If the order of \( L/\text{span}\{r, s\} \) is 2, \( 8 \leq (s, s) \in 4 + 8\mathbb{Z} \).
If the order of \( L/\text{span}\{r, s\} \) is 4, \( 28 \leq (s, s) \in 28 + 32\mathbb{Z} \).

Proof. Let \( x \) be in a nontrivial coset of \( N := \text{span}\{r, s\} \) in \( L \).
If \( (x, x) \in 2 + 4\mathbb{Z} \), \( 2x = pr + qs \), where \( p \) is odd. We have \( (x, x) \geq 6 \), \( p \) odd and \( q \neq 0 \). Therefore, \( (2x, 2x) \in 8\mathbb{Z} \), \( 24 \leq 4p^2 + (s, s)q^2 \), whence \( q \) is odd and \( (s, s) \in 4 + 8\mathbb{Z} \).

If \( (x, x) \in 1 + 2\mathbb{Z} \), \( (4x, 4x) \in 32\mathbb{Z} \). We have \( (x, x) \geq 6 \), whence \( (4x, 4x) \geq 96 \). If we write \( 4x = pr + qs \), we have \( 4p^2 + q^2(s, s) \in 32\mathbb{Z} \). Since \( p \) is odd, \( p^2 \in 1 + 8\mathbb{Z} \) and \( 4p^2 \in 4 + 32\mathbb{Z} \). Since \( (s, s) \) is even, \( q \) is odd, \( q^2 \in 1 + 8\mathbb{Z} \) and \( (s, s) \in 4 + 8\mathbb{Z} \). It follows that \( \frac{1}{4}q^2(s, s) \in 7 + 8\mathbb{Z} \) whence \( \frac{1}{4}(s, s) \in 7 + 8\mathbb{Z} \). \( \square \)

5 About idempotents in small dimensional algebras

We can derive a lot of information about the automorphism group of a vertex operator algebra by restricting to low degree homogeneous pieces. For the \( V_L^+ \) problem, the degree 2 piece and its product \( x, y \mapsto x_1y \) give an algebra which is useful to study. Here, for \( \text{rank}(L) = 2 \), we concentrate on some commutative algebras of dimension around 5. Commutativity of \( (V_L^+, 1^{st}) \) is implied if \( L_1 = \emptyset \), which is so for \( b \in \{0, \pm 1, \pm 2\} \) as in 4.4.

It does not seem advantageous to give particular values to \( b \) most of the time, so we keep it as an unspecified constant in case these arguments might be a model for future work. In the present work, we shall note limits on \( b \), as needed.

Notation 5.1. Let \( S \) be the Jordan algebra of degree 2 symmetric matrices and suppose that \( A \) is a commutative 5 dimensional algebra of the form \( A = S \oplus \mathbb{C}v_r \oplus \mathbb{C}v_s \).

Suppose that \( v_r \times v_s = 0 \) and that the notations of Appendix: Algebraic rules apply here, with the usual inner products and algebra product. Let \( w = p + c_r v_r + c_s v_s \) be
an idempotent. Suppose also that $t$ is a norm 4 vector orthogonal to $r$. Let $a_1, a_2, a_3$ be scalars so that $p = a_1r^2 + a_2rt + a_3t^2$.

**Remark 5.2.** We note that the basis $r, s$ of $H$ has dual basis $r^*, s^*$, where $r^* = \frac{1}{16-b^2}(4r - bs)$ and $s^* = \frac{1}{16-b^2}(4s - br)$. The identity of $A$ is $\frac{1}{4}\frac{1}{16-b^2}(rr^* + ss^*) = \frac{1}{4}\frac{1}{16-b^2}(4r^2 + 4s^2 - 2brs)$.

**Notation 5.3.** If $w$ is an element of $A$, write $w = p+q$ for $p \in S^2H$ and $q \in \mathbb{C}v_r \oplus \mathbb{C}v_s$. Call the element $\bar{w} := p - q$ the conjugate element. The components $p, q$ are called the $P$-part and the $Q$-part of $w$. Extend this notation to subscripted elements: $w_i = p_i + q_i, \bar{w}_i = p_i - q_i$, for indices $i$.

**Remark 5.4.** In 5.3, $q^2 \in S^2H$ since $v_r \times v_s = 0$. Also $w = p + q$ is an idempotent if and only if $p = p^2 + q^2$ and $q = 2p \times q$. Therefore, $w = p + q$ is an idempotent if and only if the conjugate $p - q$ is an idempotent.

**Lemma 5.5.** Suppose that $w_1$ and $w_2$ are idempotents and their sum is an idempotent. Then $w_1 \times w_2 = 0$ and $(w_1, w_2) = 0$.

**Proof.** We have $(w_1 + w_2)^2 = w_1^2 + 2w_1 \times w_2 + w_2^2$, whence $w_1 \times w_2 = 0$. Also, $(w_1, w_2) = (w_1^2, w_2) = (w_1, w_1 \times w_2) = 0$. □

**Definition 5.6.** Throughout this article, an idempotent is not zero or the identity, unless the context clearly allows the possibility. We call an idempotent $w$ of *type 0*, 1, 2, respectively, if it has $Q$-part which is 0, is a multiple of $v_r$ or $v_s$, or is not a multiple of either $v_r$ or $v_s$.

**Lemma 5.7.** Then (i) $r^2 \times s^2 = 4brs, r^2 \times r^2 = 16r^2, s^2 \times s^2 = 16s^2, rs \times rs = 4r^2 + 4s^2 + 2brs; x^2 \times v_r = (x, r)^2v_r = \frac{1}{2}(x^2, r^2)v_r; r^2 \times rs = 8rs + 2b^2r^2, s^2 \times rs = 8rs + 2b^2s^2$; also $v_r \times v_r = r^2, v_r \times v_s = 0, v_s \times v_s = s^2$.

(ii) $(r^2, r^2) = 32 = (s^2, s^2), (r^2, s^2) = 2b^2, (rs, rs) = 16 + b^2, (rs, r^2) = 8b = (rs, s^2)$ and $(v_r, v_r) = 2 = (v_s, v_s)$ and $(v_r, v_s) = 0$.

**Proof.** See the Appendix (and take $a = d = 4, b \neq 0, \pm 2$). □

### 5.1 Idempotents of type 0

**Lemma 5.8.** These are just idempotents in the Jordan algebra of symmetric matrices. They are ordinary idempotent matrices which are symmetric. Up to conjugacy by orthogonal transformation, they are diagonal matrices with diagonal entries only 1 and 0.
5.2 Idempotents of type 1

**Notation 5.9.** The next few results apply to the case of an idempotent of type 1, i.e., the form \( w = p + c_r v_r \), where \( c_r \neq 0 \). In such a case, \( w = w^2 = p^2 + c_r^2 r^2 + c_r (p, r^2) v_r \) (see the Appendix: Algebraic Rules). From \( c_r \neq 0 \), we get \((p, r^2) = 1\). We continue to use the notation of 5.1.

**Lemma 5.10.** Suppose that \( c_r \neq 0 \) and \( c_s = 0 \). We have
\[
a_1 = 16a_1^2 + 4a_2^2 + c_r^2; \quad a_2 = 16a_2(a_1 + a_3); \quad a_3 = 16a_3^2 + 4a_2^2 \quad \text{and} \quad (p, r^2) = 1.
\]

**Proof.** Compute \( p + c_r v_r = w = w^2 = p^2 + c_r^2 r^2 + c_r (p, r^2) v_r \) (see 5.7) and expand in the basis \( r^2, rt, t^2, v_r \). □

**Corollary 5.11.** \( a_1 = \frac{1}{32} \).

**Proof.** We have \( 1 = (p, r^2) = a_1 (r^2, r^2) = 32a_1 \), whence \( a_1 = \frac{1}{32} \). □

**Lemma 5.12.** Suppose that \( c_r \neq 0 \) and \( c_s = 0 \). Then
\[
(A1) \quad a_1 = 16a_1^2 + 4a_2^2 + c_r^2; \\
(A2) \quad a_2 = 16a_2(a_1 + a_3); \quad \text{and} \\
(A3) \quad a_3 = 16a_3^2 + 4a_2^2.
\]

**Proof.** Compute \( p^2 = (16a_1^2 + 4a_2^2)r^2 + 16(a_1a_2 + a_3a_2)rt + (16a_3^2 + 4a_2^2)t^2 \) and use \( w = w^2 = p^2 + c_r^2 r^2 + c_r v_r \). □

**Lemma 5.13.** Suppose that \( c_r \neq 0 \) and \( c_s = 0 \). If \( a_2 = 0 \), then \( a_3 \in \{0, \frac{1}{16}\} \) and \( c_r = \pm \frac{1}{8} \).

**Proof.** We deduce from (A3) that \( a_3 = 16a_3^2 \), then \( c_r = \pm \frac{1}{8} \). □

**Lemma 5.14.** Suppose that \( c_r \neq 0 \) and \( c_s = 0 \). Then \( a_2 = 0 \).

**Proof.** If \( a_2 \neq 0 \), then from (A2), \( 1 = 16(a_1 + a_3) \) and we get \( a_3 = \frac{1}{32} \). Next, use (A3) to get \( \frac{1}{64} = 4a_2^2 \). Finally use (A1) to get \( c_r = 0 \). □

**Theorem 5.15.** Assume that \( c_r \neq 0 \) and \( c_s = 0 \). Then
\[
(i) \quad a_1 = \frac{1}{32}, \quad a_2 = 0, \quad c_r = \pm \frac{1}{8}; \quad \text{and} \\
(ii) \quad \text{either} \quad (w, w) = \frac{1}{16} \quad \text{and} \quad a_3 = 0; \quad \text{or} \quad (w, w) = \frac{3}{16} \quad \text{and} \quad a_3 = \frac{1}{16}.
\]

All of the above cases occur. If an idempotent occurs, so does its complementary idempotent.

**Proof.** This is a summary of preceding results. □
Lemma 5.16. If $w$ is an idempotent of type 1, then

(i) if $(w, w) = \frac{1}{16}$, the eigenvalues for $ad(w)$ are 1, 0, 0, $\frac{1}{4}$, $\frac{\beta^2}{32}$. Eigenvectors for these respective eigenspaces are $w, 1 - w, t^2, rt, vs$;

(ii) if $(w, w) = \frac{3}{16}$, the eigenvalues for $ad(w)$ are 0, 1, 1, $\frac{3}{4}, 1 - \frac{\beta^2}{32}$. Eigenvectors for these respective eigenspaces are $w, 1 - w, t^2, rt, vs$.

If $\frac{\beta^2}{32} \neq 0, 1, \frac{1}{4}, \frac{3}{4}$, the multiplicities of 0 and 1 are 2 and 1 in case (i) and 1 and 2 in case (ii).

Proof. Straightforward calculation. Note that $\frac{\beta^2}{32} \neq 0, 1, \frac{1}{4}, \frac{3}{4}$ follows if $b$ is rational \(\square\)

Corollary 5.17. If $w$ is a type 1 idempotent and is the sum of two nonzero idempotents, $w_1, w_2$, then $w$ has the form $\frac{1}{32}r^2 + \frac{1}{16}t^2 \pm \frac{1}{8}v_r$ and $w_1, w_2$ are, up to order, $\frac{1}{32}r^2 \pm \frac{1}{8}v_r$ and $\frac{1}{16}t^2$.

Proof. If $w$ is such a sum, each $w_i$ is in the 1-eigenspace of $ad(w)$, which must be more than 1-dimensional. This means that $w$ has norm $\frac{3}{16}$ and one of the $w_i$, say for $i = 1$, has type 1 and $Q$-part $\pm \frac{1}{8}v_r$. Therefore, $w_2$ has type 0, whence norm $\frac{1}{8}$. This means that $w_1$ has norm $\frac{1}{16}$ and so we know that $w_1$ has shape $\frac{1}{32}r^2 \pm \frac{1}{8}v_r$ and $w = \frac{1}{16}t^2$. \(\square\)

5.3 Idempotents of type 2

Hypothesis 5.18. We assume in this subsection that the parameter $b \neq 0, \pm 2, \pm 3$ (which means $b = \pm 1$). Then the algebra $(V_2, 1^{st})$ is commutative since $V_1 = 0$.

Notation 5.19. $p = c(r^2 + s^2) + drs$, $v = c_rv_r + c_sv_s$.

Lemma 5.20. If $c_r$ and $c_s$ are nonzero, then there are at most 8 possibilities for $w$.

In more detail, there are at most two values of $c$ (and, correspondingly, of $d$). We have $c_r^2 = c_s^2$ and this common value depends on $c$ (or on $d$).

Proof. Compute $p + c_rv_r + c_sv_s = w = w^2 = p^2 + c_r^2r^2 + c_s^2s^2 + c_r(p, r^2)v_r + c_s(p, s^2)v_s$. Since $c_r$ and $c_s$ are nonzero, $(p, r^2) = 1 = (p, s^2)$.

Since $(r^2, r^2) = 32 = (s^2, s^2)$, $(rs, r^2) = 8b = (rs, s^2)$ and $(r^2, s^2) = 16 + b^2$, we have $p = c(r^2 + s^2) + drs$. For some scalars, $c, d$. The previous paragraph then implies that $1 = (32 + 2b^2)c + 8bd$. Since $b \neq 0$,

\[(e1) \quad d = \frac{1}{8b}(2c(32 + 2b^2) - 1)\]

is a linear expression in $c$. 

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Now, \( p^2 = (16c^2 + 4d^2)(r^2 + s^2) + (2bc^2 + 2bd^2)rs \) and so \( w^2 = (16c^2 + 4d^2 + c^2_r)r^2 + (16c^2 + 4d^2 + c^2_s)s^2 + (2bc^2 + 2bd^2)rs + c_rv_r + c_sv_s \). It follows that \( c^2_r = c^2_s \).

We compare coefficients of \( r^2 \) and get
\[
(c2) \quad c = 16c^2 + 4d^2 + c^2_r.
\]

We compare coefficients of \( rs \) and get
\[
(c3) \quad d = 2bc^2 + 2bd^2.
\]

Since \( d \) is a linear expression in \( c \), \( c \) satisfies a quadratic equation, depending on \( b \) but not \( c^2_r \). The degree of this equation really is 2 since \( b \neq 0 \) real implies that the top coefficient is nonzero.

It follows that the ordered pair \( d, c \) has at most two possible values. For each, there is a unique value for \( c^2_r \), hence at most two possible values for \( c_r \) (and the same two for \( c_s \)). Therefore there are at most eight idempotents of type 2. □

Lemma 5.21. \( c \neq 0 \) and \( d \neq 0 \).

**Proof.** Suppose that \( c = 0 \). We then have \( p = drs, d = \frac{-1}{8b} \). On the other hand, since \( w = drs + v \) is an idempotent, the coefficient for \( w^2 \) at \( rs \) is \( d = 8bc^2 + 2bd^2 = 2bd^2 = 2bd^2 \), which implies that \( 1 = 2bd \). This is incompatible with \( d = \frac{-1}{8b} \).

If \( d = 0 \), equation (e3) implies that \( c = 0 \), which is false. □

Lemma 5.22. If \( w \) is a type 2 idempotent, \( w = c(r^2 + s^2) + drs + c_rv_r + c_sv_s \) and \( 1 - w \) is the complementary idempotent, expanded similarly as \( 1 - w = c'(r^2 + s^2) + d'rs + c'_rv_r + c'_sv_s \), then \( c' = -c_r \neq c_r, \ c'_s = -c_s, \ c \neq c' \) and \( d \neq d' \). In particular, in the notation of 5.20, the function \( c \mapsto c^2_r \) is 2-to-1 and so only one value of \( c^2_r \) occurs for type 2 idempotents.

**Proof.** If it were true that \( c = c' \), then \( w = \frac{1}{2}I + v \) and \( 1 - w = \frac{1}{2}I - v \). Since these are idempotents, \( v^2 = \frac{1}{2}I \). However, this is impossible as \( b \neq \pm 2 \) implies that \( v^2 \) is a multiple of \( r^2 + s^2 \) and \( I \) is not a linear combination of \( r^2, s^2 \) for \( b \neq 0 \) (see 5.2). □

### 5.4 Sums of idempotents

**Hypothesis 5.23.** We continue to take \( b = \pm 1 \). Results of the previous subsection apply.

In the arguments in this section, we allow the symbol \( b \) to be any odd integer, though the lattice is positive definite only for \( b = \pm 1 \).
Lemma 5.24. Suppose that $w_1, w_2$ are two idempotents of type 1. If $w_1 + w_2$ is an idempotent, then $w_1 + w_2$ does not have type 1 or type 2.

Proof. We eliminate the sum having type 1 with Lemmas 5.17. To eliminate a sum having type 2, we note that for type 1 idempotents, we have $a_2 = 0$ by 5.15, whereas $d \neq 0$ for type 2, by 5.21. □

Lemma 5.25. If $w_1, w_2$ are idempotents of type 2 and not complementary, their sum is not an idempotent.

Proof. Assume that the sum $w$ is an idempotent. From 5.24, the sum has type 0, so has the form $\frac{1}{16}u^2$, for some vector $u \in H$ of norm 4. The eigenvalues of $ad(u)$ are $1, 0, \frac{1}{2}$ and $\frac{1}{16}(u, r)^2, \frac{1}{16}(u, s)^2$, with respective eigenvectors $u^2, 1 - u^2, \frac{1}{2}u'u, v_r, v_s$, where $u'$ spans the orthogonal of $u$ in $H$.

Now, $w_1, w_2$ are linearly independent (or else they are equal, which is impossible). This means that the eigenvalue 1 has multiplicity at least 2. So, at least one of $(u, r)^2, (u, s)^2$ is 16. Since $w_1, w_2$ lie in the 1-eigenspace of $ad(u)$ and both $w_i$ have type 2, both these square norms must be 16, i.e., $m := (u, r) = \pm 4$ and $n := (u, s) = \pm 4$. Since $r, s$ form a basis and the form is nonsingular, this forces $u = mr^* + ns^*$, where $r^*, s^*$ is the dual basis. We have $4 = (u, u) = 16(r^*, r^*) + 2mn(r^*, s^*) + 16(s^*, s^*)$.

The right side is $\frac{1}{16-b^2}[16(4r - bs, 4r - bs) + 2mn(4r - bs, 4s - br) + 16(4s - br, 4s - br)]$.

Since $b$ is an odd integer, the above rational number in reduced form clearly has numerator divisible by 16, so does not equal 4, a contradiction. □

Lemma 5.26. The sum of a type 1 and type 2 idempotent is not an idempotent.

Proof. Assume that $w := w_1 + w_2$ is an idempotent. Obviously it does not have type 0. By 5.17, it does not have type 1.

We conclude that $w$ has type 2. However, the coefficients of $w$ at $r^2$ and $s^2$ must be equal for type 2, a contradiction since this forces the $P$-part if the type 1 idempotent to be 0. □

Corollary 5.27. The only idempotents which are a proper summand of some non-trivial idempotent are the ones of type 1 and norm $\frac{1}{16}$. There are 4 such and they come in orthogonal pairs, which are just pair of idempotents and their conjugates.

Corollary 5.28. Aut($A$) is a dihedral group of order 8.

Proof. The automorphism group preserves and acts faithfully on the set $J$ of type 1 idempotents of norm $\frac{1}{16}$, the complete set of idempotents which are proper summands of proper idempotents, and furthermore preserves the partition defined by
orthogonality. The orthogonal in $A$ of the nonsingular subspace $\text{span}(J)$ is spanned by $v := r^2 + s^2 - \frac{16 + b^2}{4b} rs$. We claim that if an automorphism acts trivially on $\text{span}(J)$, it acts trivially on $A$. This is so because $rs \in \text{span}\{r^2 \times s^2, r^2, s^2\}$ and $\{rs\} \cup J$ spans $A$.

This proves that the automorphism group of $A$ embeds in a dihedral group of order 8. This embedding is an isomorphism onto since the LVOA$^+$ group embeds in $\text{Aut}(A)$.

Proposition 5.29. $\text{Aut}(V_L^+)$ is just the LVOA$^+$-group, isomorphic to $\text{Dih}_8$.

Proof. In this case we have $(V_L^+)_2 = M(1)_2^+$. Thus any automorphism of $V_L^+$ preserves $M(1)_2^+$. Now use 3.3.

6 Automorphism group of $V_L^+$ with rank$(L) = 2$

In this section, we assume that the rank of $L$ is equal to 2. If $L_1 = L_2 = \emptyset$, the automorphism group of $V_L^+$ was determined in Proposition 3.4. So in this section we assume that $L_1$ or $L_2$ is not empty.

6.1 $L_1 = \emptyset$ and rank$(L_2) = 2$; $b = 0$.

Note that $L$ is generated by $L_2$. We will discuss the automorphism group according to the value $b$ in the Gram matrix $G$ (see 4.4).

First we assume that $b = 0$ in the Gram matrix $G$. Then $L \cong \sqrt{2}L_{A_1} \perp \sqrt{2}L_{A_1}$, where $L_{A_1}$ is the root lattice of type $A_1$. Let $L = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ with $(\alpha_i, \alpha_j) = 4\delta_{ij}$ for $i, j = 1, 2$. Set

$$\omega_1 = \frac{1}{16} \alpha_1 (-1)^2 + \frac{1}{4} (e^{\alpha_1} + e^{-\alpha_1}),$$

$$\omega_2 = \frac{1}{16} \alpha_2 (-1)^2 - \frac{1}{4} (e^{\alpha_2} + e^{-\alpha_2}).$$

We also use $\alpha_2$ to define $\omega_3$ and $\omega_4$ in the same fashion. Then $\omega_i$ for $i = 1, 2, 3, 4$ are commutative Virasoro vectors of central charge $\frac{1}{2}$ (see [DMZ] and [DGH]). It is well-known that $(V_L^+)_2$ is a commutative (nonassociative) algebra under $u \times v = u_1 v$ since the degree 1 part is 0 (cf. [FLM]). Let $X$ be the span of $\omega_i$ for all $i$.

Lemma 6.1. If $u \in (V_L)_2$ is a Virasoro vector of central charge $1/2$ then $u = \omega_i$ for some $i$. 

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Proof. The space $(V_L)_2$ is 5-dimensional with a basis
\[ \{\omega_1, \omega_2, \omega_3, \omega_4, \alpha_1(-1)\alpha_2(-1)\} \].
Let \( u = \sum_{i=1}^{4} c_i\omega_i + x\alpha_1(-1)\alpha_2(-1) \in (V_L)_2 \) be a Virasoro vector of central charge \( 1/2 \). Then \( u \times u = 2u \). Note that \( \omega_i \times \omega_j = \delta_{i,j}2\omega_i \) for \( i, j = 1, 2, 3, 4 \), \( \omega_i \times \alpha_1(-1)\alpha_2(-1) = \frac{1}{2}\alpha_1(-1)\alpha_2(-1) \) and \( \alpha_1(-1)\alpha_2(-1) \times \alpha_1(-1)\alpha_2(-1) = 4\alpha_1(-1)^2 + 4\alpha_2(-1)^2 \). So we have a nonlinear system
\[
2c_i = 2c_i^2 + 32x^2, \quad i = 1, 2, 3, 4
\]
\[
2x = \sum_{i=1}^{4} xc_i.
\]
If \( x \neq 0 \) then \( \sum_{i=1}^{4} c_i = 2 \) and \( 2 = \sum_{i=1}^{4} c_i^2 + 64x^2 \).
Since the central charge of \( u \) is \( 1/2 \) we have
\[
\frac{1}{4} = u_3u = \sum_{i=1}^{4} \frac{c_i^2}{4} + 16x^2
\]
and \( 1 = \sum_{i=1}^{4} c_i^2 + 64x^2 \). This is a contradiction. So \( x = 0 \). This implies that \( c_i = 0, 1 \) and \( u = \omega_i \) for some \( i \). \( \square \)

By Lemma 6.1, any automorphism \( \sigma \) of \( V^+_L \) induces a permutation of the four \( \omega_i \).
It is known from [FLM] that \( (V^+_L)_2 \) has a nondegenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \) given by \( \langle u, v \rangle = u_3v \) for \( u, v \in (V^+_L)_2 \). The orthogonal complement of \( X \) in \( (V^+_L)_2 \) with respect to the form is spanned by \( \alpha_1(-1)\alpha_2(-1) \). Thus \( \sigma\alpha_1(-1)\alpha_2(-1) = \lambda\alpha_1(-1)\alpha_2(-1) \) for some nonzero constant \( \lambda \). Since \( \alpha_1(-1)\alpha_2(-1) \times \alpha_1(-1)\alpha_2(-1) = \alpha_1(-1)^2 + \alpha_2(-1)^2 \) which is a multiple of the Virasoro element \( \omega \). This shows that \( \lambda = \pm 1 \).

On the other hand, \( V^+_L \cong V^+_{\sqrt{2}L_1A_1} \otimes V^+_{\sqrt{2}L_1A_1} \oplus V^-_{\sqrt{2}L_1A_1} \otimes V^-_{\sqrt{2}L_1A_1} \). By Corollary 3.3 of [DGH],
\[
V^+_L \cong L(\frac{1}{2}, 0)^{\otimes 4} \oplus L(\frac{1}{2}, \frac{1}{2})^{\otimes 4}.
\]
So if the restriction of \( \sigma \) to \( X \) is identity, then the action of \( \sigma \) on \( V^+_{\sqrt{2}L_1A_1} \otimes V^+_{\sqrt{2}L_1A_1} \) is trivial and on \( V^-_{\sqrt{2}L_1A_1} \otimes V^-_{\sqrt{2}L_1A_1} \) is \( \pm 1 \). Indeed, there is automorphism \( \tau \) of \( V^+_L \) such that \( \tau \) acts trivially on \( V^+_{\sqrt{2}L_1A_1} \otimes V^+_{\sqrt{2}L_1A_1} \) and acts as \( -1 \) on \( V^-_{\sqrt{2}L_1A_1} \otimes V^-_{\sqrt{2}L_1A_1} \) by the fusion
role for $V^+_{\sqrt{2}L_{A_1}}$ (see [ADL]). As $V^+_{\sqrt{2}L_{A_1}} \otimes V^+_{\sqrt{2}L_{A_1}}$ is generated by $\omega_i$ for $i = 1, 2, 3, 4$, any automorphism preserves $V^+_{\sqrt{2}L_{A_1}} \otimes V^+_{\sqrt{2}L_{A_1}}$ and its irreducible module $V^-_{\sqrt{2}L_{A_1}} \otimes V^-_{\sqrt{2}L_{A_1}}$ (cf. [DM1]). As a result, $\langle \tau \rangle$ is a normal subgroup of $\text{Aut}(V^+_L)$ isomorphic to $\mathbb{Z}_2$.

Next we show how $\text{Sym}_4$ can be realized as a subgroup of $\text{Aut}(V^+_L)$ by showing that any permutation $\sigma \in \text{Sym}_4$ gives rise to an automorphism of $V^+_L$. But it is clear that $\text{Sym}_4$ acts on $V^+_L$ by permuting the tensor factors. In order to see that $\text{Sym}_4$ acts on $V^+_L$ as automorphisms, it is enough to show that $\sigma(Y(u, z)v) = Y(\sigma u, z)\sigma v$ for $\sigma \in \text{Sym}_4$ and $u, v \in V^+_L$. There are 4 different ways to choose $u, v$. We only discuss the case that $u, v \in L(\frac{1}{2}, \frac{1}{2}) \otimes 4$ since the other cases can be dealt with in a similar fashion. Let $u = u^1 \otimes u^2 \otimes u^3 \otimes u^4$ and $v = v^1 \otimes v^2 \otimes v^3 \otimes v^4$ where $u_i, v_i$ are tensor factors in the $i$-th $L(\frac{1}{2}, \frac{1}{2})$. Let $Y$ be a nonzero intertwining operator of type $(L(\frac{1}{2}, 0))$. Then, up to a constant,

$$Y(u, z)v = Y(u_1, z)v_1 \otimes Y(u_2, z)v_2 \otimes Y(u_3, z)v_3 \otimes Y(u_4, z)v_4$$

(see [DMZ]). Since $\sigma$ is a permutation, it is trivial to verify that $\sigma(Y(u, z)v) = Y(\sigma u, z)\sigma v$.

So we have proved the following:

**Proposition 6.2.** If $b = 0$ in the Gram matrix $G$ then $L \cong \sqrt{2}L_{A_1} \times \sqrt{2}L_{A_1}$ and $\text{Aut}(V^+_L) \cong \text{Sym}_4 \times \mathbb{Z}_2$.

**Remark 6.3.** Here is a different proof that $\text{Aut}(V^+_L)$ contains a copy of $\text{Sym}_4 \times \mathbb{Z}_2$, using the theory of finite subgroups of Lie groups. Our lattice $L$ lies in $M \cong L_{A_2}$. Take $V_M$, which is a lattice VOA. By [DN1], $V_M$ has automorphism group isomorphic to $\text{PSL}(2, \mathbb{C}) \wr 2$. In $\text{PSL}(2, \mathbb{C})$, there is up to conjugacy a unique four group and its normalizer is isomorphic to $\text{Sym}_4$. Correspondingly, in $\text{PSL}(2, \mathbb{C}) \wr 2$ there is a subgroup isomorphic to $\text{Sym}_4 \wr 2$. In this, take a subgroup $H$ of the form $2^4: [\text{Sym}_3 \times 2]$. Let $t$ be an involution of $H$ which maps to the central involution of $H/O_2(H) \cong \text{Sym}_3 \times 2$ and take $R := C_{O_2(H)}(t) \cong 2^4$. Take the fixed points $V^R_M$. We have that $V^R_M$ is isomorphic to our $V^+_L$. So, $V^+_L$ gets an action of $H/R \cong 2^4: [\text{Sym}_3 \times 2] \cong \text{Sym}_4 \times 2$.

### 6.2 $L_1 = \emptyset$ and rank($L_2$) = 2; $b = 2$.

Next we assume that $b$ in the Gram matrix is 2. Then $L \cong \sqrt{2}L_{A_2}$. Then $L = \mathbb{Z} \alpha_1 + \mathbb{Z} \alpha_2$ with $(\alpha_1, \alpha_i) = 4$ and $(\alpha_1, \alpha_2) = 2$. As before we define $\omega_1, \omega_2$ by using $\alpha_1$, $\omega_3, \omega_4$ by using $\alpha_2$ and $\omega_5, \omega_6$ by using $\alpha_1 + \alpha_2$. Then $\omega_i$, for $i = 1, \ldots, 6$ form a basis of $(V^+_L)_2$. 

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Lemma 6.4. If \( u \in (V^+_L)_2 \) is a Virasoro vector of central charge 1/2, then \( u = \omega_i \) for some \( i \).

**Proof.** *First proof.* (There will be a second proof in the next section.)

Let \( u = \sum_{i=1}^{6} c_i \omega_i \) for some \( c_i \in \mathbb{C} \). Then \( u \) is a Virasoro vector of central charge 1/2 if and only if \( (u, u) = 1/4 \) and \( u \times u = 2u \). Note that

\[
(\omega_i, \omega_i) = 1/4, (\omega_{2j-1}, \omega_{2j}) = 0, \quad 1 \leq i \leq 6, j = 1, 2, 3
\]

\[
(\omega_1, \omega_k) = (\omega_2, \omega_k) = \frac{1}{32}, \quad k = 3, 4, 5, 6.
\]

So we have

\[
(u, u) = \frac{1}{4} \sum_{i=1}^{6} c_i^2 + \frac{1}{16} \sum_{j=1,2} \sum_{j<k\leq3} (c_{2j-1}c_{2k-1} + c_{2j-1}c_{2k} + c_{2j}c_{2k-1} + c_{2j}c_{2k}) = \frac{1}{4}.
\]

In order to compute \( u \times u \) we need the following multiplication table in \((V^+_L)_2\) :

\[
\omega_{2i-1} \times \omega_{2i} = 0, \quad i = 1, 2, 3
\]

\[
\begin{align*}
\omega_1 \times \omega_3 &= \frac{1}{4}(\omega_1 + \omega_3 - \omega_6), & \omega_2 \times \omega_3 &= \frac{1}{4}(\omega_2 + \omega_3 - \omega_5) \\
\omega_1 \times \omega_4 &= \frac{1}{4}(\omega_1 + \omega_4 - \omega_5), & \omega_2 \times \omega_4 &= \frac{1}{4}(\omega_2 + \omega_4 - \omega_6) \\
\omega_1 \times \omega_5 &= \frac{1}{4}(\omega_1 + \omega_5 - \omega_4), & \omega_2 \times \omega_5 &= \frac{1}{4}(\omega_2 + \omega_5 - \omega_3) \\
\omega_1 \times \omega_6 &= \frac{1}{4}(\omega_1 + \omega_6 - \omega_3), & \omega_2 \times \omega_6 &= \frac{1}{4}(\omega_2 + \omega_6 - \omega_4) \\
\omega_3 \times \omega_5 &= \frac{1}{4}(\omega_3 + \omega_5 - \omega_2), & \omega_4 \times \omega_5 &= \frac{1}{4}(\omega_4 + \omega_5 - \omega_1) \\
\omega_3 \times \omega_6 &= \frac{1}{4}(\omega_3 + \omega_6 - \omega_1), & \omega_2 \times \omega_6 &= \frac{1}{4}(\omega_2 + \omega_6 - \omega_2).
\end{align*}
\]

Then \( u \times u = 2u \) if and only if

\[
\begin{align*}
c_1^2 + \frac{1}{4}(c_1c_3 + c_1c_4 + c_1c_5 + c_1c_6 - c_3c_6 - c_4c_5) &= c_1 \\
c_2^2 + \frac{1}{4}(c_2c_3 + c_2c_4 + c_2c_5 + c_2c_6 - c_3c_5 - c_4c_6) &= c_2 \\
c_3^2 + \frac{1}{4}(c_1c_3 + c_2c_3 + c_3c_5 + c_3c_6 - c_1c_6 - c_2c_5) &= c_3
\end{align*}
\]
\[ c_1^2 + \frac{1}{4}(c_1c_4 + c_2c_4 + c_4c_5 + c_4c_6 - c_1c_5 - c_2c_6) = c_4 \]
\[ c_5^2 + \frac{1}{4}(c_1c_5 + c_2c_5 + c_3c_5 + c_4c_6 - c_1c_4 - c_2c_3) = c_5 \]
\[ c_6^2 + \frac{1}{4}(c_1c_6 + c_2c_6 + c_3c_6 + c_4c_6 - c_1c_3 - c_2c_4) = c_6. \]

There are exactly 6 solutions to this linear system: \( c_i = 1 \) and \( c_j = 0 \) if \( j \neq i \) where \( i = 1, ..., 6 \). We thank Harm Derksen for obtaining this result with the MacCauley software package. This finishes the proof of the lemma. \( \square \)

**Proposition 6.5.** If \( b = 2 \) in the Gram matrix, then \( L \cong \sqrt{2} L A_2 \) and \( \text{Aut}(V_L^+) \) is the LVOA\(^+\)-group.

**Proof.** First note that the Weyl group acts on \( L \), preserving and acting as \( \text{Sym} 3 \) on the set \( \{\{\pm \alpha_1\}, \{\pm \alpha_2\}, \{\pm (\alpha_1 + \alpha_2)\}\} \).

Now let \( \sigma \in \text{Aut}(V_L^+) \). Set \( X_i = \{\omega_{2i-1}, \omega_{2i}\} \) for \( i = 1, 2, 3 \). Then \( X_i \) are the only orthogonal pairs in \( X = X_1 \cup X_2 \cup X_3 \). Since \( \sigma X = X \) we see that \( \sigma \) induces a permutation on the set \( \{X_1, X_2, X_3\} \).

The above shows that \( O(\hat{L}) \) induces \( \text{Sym} 3 \) on this 3-set. We may therefore assume that \( \sigma \) preserves each \( X_i \). In this case \( \sigma \) acts trivially on \( \alpha_1(-1)^2, \alpha_2(-1)^2, (\alpha_1 + \alpha_2)(-1)^2 \). That is, \( \sigma \) acts trivially on the subVOA they generate, which is isomorphic to \( M(1)^+ \). As a result, \( \sigma \) is in the LVOA\(^+\)-group. \( \square \)

### 6.3 Alternate proof for \( b = 2 \)

The system of equations in the variables \( c_i \) which occurred in the proof of 6.4 can be replaced by an equivalent system 6.7 which looks more symmetric. The old system was solved with software package MacCauley but not with Maple. The new system was solved with Maple and gives the same result as before.

**Notation 6.6.** Let \( r \) and \( s \) be independent norm 4 elements so that \( t := -r - s \) has norm 4. Let \( w \) be an idempotent \( w = p + q \), where \( p = ar^2 + bs^2 + ct^2 \) and \( q = dw + ev_s + ev_t \) which satisfies \( (w, w) = \frac{1}{16} \). Since \( (L, L) \leq 2\mathbb{Z} \), we may and do assume that the epsilon-function is identically 1. It follows that \( v_r \times v_s = v_t \) and similarly for all permutations of \( \{r, s, t\} \).

**Lemma 6.7.** From \( w^2 = w \), we have equations

\[(e1) \quad a = 16a^2 + 4ab + 4ac - 4bc + d^2 \]
\((e2)\) \quad b = 16b^2 + 4bc + 4ba - 4ac + e^2

\((e3)\) \quad c = 16c^2 + 4cq + 4cb - 4ab + f^2

\((e4)\) \quad d = 2d(16a + 4b + 4c) + 2ef

\((e5)\) \quad e = 2e(4a + 16b + 4c) + 2df

\((e6)\) \quad f = 2f(4a + 4b + 16c) + 2de

and from \((w, w) = \frac{1}{16}\), we get the equation

\((e7)\) \quad \frac{1}{16} = 32(a^2 + b^2 + c^2) + 16(ab + ac + bc) + 2(d^2 + e^2 + f^2).

**Proof.** Straightforward from Appendix: Algebraic rules. □

**Proposition 6.8.** There are just 6 solutions \((a, b, c, d, e, f) \in \mathbb{C}^6\) to the equations \((e1), \ldots, (e7)\). They are \((\frac{1}{32}, 0, 0, \frac{1}{8}, 0, 00), (\frac{1}{32}, 0, 0, -\frac{1}{8}, 0, 00)\) and ones obtained from these by powers of the permutation \((abc)(def)\).

**Proof.** This follows from use of the `solve` command in the software package Maple. □

**Remark 6.9.** If we omit \((e7)\), there are infinitely many solutions with \(d = e = f = 0\). The reason is that the Jordan algebra of symmetric degree 2 matrices has infinitely many idempotents. It seems possible that the system in Lemma 6.7 could be solved by hand.

### 6.4 \(L_1 = \emptyset\) and \(\text{rank}(L_2) = 2\); \(b = 1\).

We now deal with the cases \(b = 1\) in the Gram matrix.

**Proposition 6.10.** If \(b = 1\) in the Gram matrix, then \(\text{Aut}(V_L^+)\) is the LVOA\(^+\) group.

**Proof.** By Corollary 5.27, any automorphism of \(V_L^+\) preserves \(M(1)_2^+\), the result follows from Lemma 3.3 □
7 \textbf{Aut}(V^+_L), \text{ for } L_1 = \emptyset, \text{ rank}(L_2) = 1

In this case we can assume that \( L_2 = \{2\alpha_1, -2\alpha_1\} \). Let \( \alpha_2 \in H \) such that \((\alpha_i, \alpha_j) = \delta_{i,j}\). Then \((V^+_L)_2\) is 4-dimensional with basis \( v_{2\alpha_1}, \frac{1}{2}\alpha_1(-1)^2, \frac{1}{2}\alpha_2(-1)^2, \alpha_1(-1)\alpha_2(-1) \).

\textbf{Lemma 7.1.} Any automorphism of \( V^+_L \) preserves the subspace \( S^2H \) of \((V^+_L)_2\) spanned by \( \frac{1}{2}\alpha_1(-1)^2, \frac{1}{2}\alpha_2(-1)^2, \alpha_1(-1)\alpha_2(-1) \).

\textbf{Proof.} Since Virasoro vectors of central charge 1 in \( S^2H \) span \( S^2H \), it is enough to show that any Virasoro vector of central charge 1 lies in \( S^2H \).

Let \( t = d_1 \frac{\alpha_2^2}{2} + d_2 \frac{\alpha_2^2}{2} + d_3 v_{2\alpha_1} + d_4 \alpha_1 \alpha_2 \) be a Virasoro vector of central charge 1 with \( d_3 \neq 0 \). Then we must have \( t \times t = 2t \) and \((t, t) = 1/2\). A straightforward computation shows that
\[
t \times t = d_1^2 \alpha_1^2 + d_2^2 \alpha_2^2 + d_3^2(2\alpha_1)^2 + d_4^2(\alpha_1^2 + \alpha_2^2) + 4d_1d_3 v_{2\alpha_1} + 2d_1d_4 \alpha_1 \alpha_2 + 2d_2d_4 \alpha_1 \alpha_2.
\]

This gives four equations
\[
\begin{align*}
d_1 &= d_1^2 + 4d_3^2 + d_4^2 \\
d_2 &= d_2^2 + d_4^2 \\
d_3 &= 2d_1d_3 \\
d_4 &= d_1d_4 + d_2d_4.
\end{align*}
\]

The relation \((t, t) = 1/2\) gives one more equation:
\[
\frac{1}{2} = \frac{1}{2}d_1^2 + \frac{1}{2}d_2^2 + d_3^2 + 2d_3^2.
\]

Thus
\[
1 = d_1 + d_2.
\]

Since \( d_3 \neq 0 \), \( d_1 = 1/2 \) and \( d_2 = 1/2 \). So we have
\[
\frac{1}{4} = 4d_3^2 + d_4^2, \quad \frac{1}{4} = 2d_3^2 + d_4^2.
\]

This forces \( d_3 = 0 \), a contradiction. \( \Box \)

\textbf{Proposition 7.2.} In this case, \( \text{Aut}(V^+_L) \) is the \( \text{LVOA}^+ \)-group.

\textbf{Proof.} By Corollary 5.27, any automorphism of \( V^+_L \) preserves \( M(1) \), the result follows from Lemma 3.3 \( \Box \)
8  \textit{Aut}(V_L^+), \text{ for } L_1 \neq \emptyset

Finally we deal with the case that $L_1 \neq \emptyset$. There are two cases: \text{rank}(L_1) = 2 or \text{rank}(L_1) = 1.

8.1  \text{rank}(L_1) = 2

In this case $L = L_{A_2}^1$ or $L = L_{A_2}$ because these are the only rank 2 root lattices possible and each is a maximal even integral lattice in its rational span.

8.1.1  $L$ has type $A_2^1$

If $L \cong L_{A_2}^1$ then \textit{Aut}(V_L) \cong PSL(2, \mathbb{C}) \rtimes 2 and

$$V_L^+ \cong V_{L_{A_1}}^+ \otimes V_{L_{A_1}}^+ \oplus V_{L_{A_1}}^- \otimes V_{L_{A_1}}^-.$$  

Since the connected component of the identity in \textit{Aut}(V_L) contains a lift of $-1_L$, we may assume that such a lift is in a given maximal torus, so is equal to the automorphism $e^{\pi i \beta(0)/2}$, where $\beta$ is a sum of orthogonal roots.

It follows that $V_L^+ \cong V_K$, where $K = 2L + \mathbb{Z}\beta$. The result [DN1] implies that \textit{Aut}(V_L^+) \cong \textit{Aut}(V_K)$, which is the LVOA group $\mathbb{T}_2.Dih_8$.

8.1.2  $L$ has type $A_2$

Here, $(V_L^+)_1$ is a 3-dimensional Lie algebra isomorphic to $sl(2, \mathbb{C})$. The difficult part in this case is to determine the vertex operator subalgebra generated by $(V_L^+)_1$. Let $L_{A_2} = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ such that $(\alpha_i, \alpha_i) = 2$ and $(\alpha_1, \alpha_2) = -1$. The set of roots in $L$ is $L_1 = \{ \pm \alpha_i | i = 1, 2, 3 \}$ where $\alpha_3 = \alpha_1 + \alpha_2$. The positive roots are $\{ \alpha_i | i = 1, 2, 3 \}$. The space $(V_L^+)_1$ is 3-dimensional with a basis $v_{\alpha_i}$ for $i = 1, 2, 3$ and $(V_L^-)_1$ is 5-dimensional with a basis $\alpha_1(-1), \alpha_2(-1), e^{\alpha_i} - e^{-\alpha_i}$ for $i = 1, 2, 3$. It is a straightforward to verify that $(v_{\alpha_i})_{-1}v_{\alpha_i}$ for $i = 1, 2, 3$ and $\alpha_i(-1)^2$ for $i = 1, 2, 3$ span the same space. Thus $\omega = \frac{1}{4} \alpha_1(-1)^2 + \frac{1}{12} (\alpha_1(-1) + 2\alpha_2(-1))^2$ lies in the vertex operator algebra generated by $(V_L^+)_1$.

In order to determine the vertex operator algebra generated by $(V_L^+)_1$ we need to recall the standard modules for affine algebra

$$A_{1}^{(1)} = sl(2, \mathbb{C}) = sl(2, \mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$$

(cf. [DL]). We use the standard basis $\{ \alpha, x_\alpha, x_{-\alpha} \}$ for $sl(2, \mathbb{C})$ such that

$$[\alpha, x_{\pm\alpha}] = \pm 2x_{\pm\alpha}, [x_\alpha, x_{-\alpha}] = \alpha.$$
We fix an invariant symmetric nondegenerate bilinear form on \( sl(2, \mathbb{C}) \) such that \((\alpha, \alpha) = 2\). The level \( k \) standard \( A_1^{(1)} \)-modules are parametrized by dominant integral linear weights \( \frac{1}{k} \alpha \) for \( i = 0, \ldots, l \) such that the highest weight of the \( A_1^{(1)} \)-module, viewed as a linear form on \( \mathbb{C}\alpha \oplus \mathbb{C}K \subset \mathfrak{sl}(2, \mathbb{C}) \), is given by \( \frac{i}{2} \alpha \) and the correspondence \( K \mapsto k \). Let us denote the corresponding standard \( A_1^{(1)} \)-module by \( L(k, \frac{i}{2} \alpha) \). It is well known that \( L(k, 0) \) is a simple rational vertex operator algebra and \( L(k, \frac{i}{2} \alpha) \) for \( i = 0, \ldots, k \) is a complete list of irreducible \( L(k, 0) \)-modules (cf. [DL], [FZ] and [L2]).

Note that \( \omega \), \( \alpha \), \( \beta \) and \( \omega \) act as a constant on \( \mathfrak{sl}(2, \mathbb{C}) \) with respect to the standard bilinear form. Then \( \omega' = \frac{1}{2(4k+2)} \sum_{i=1}^{3} v_i (-1)^i 2 \mathbf{1} \in V \) is the Segal-Sugawara Virasoro vector. Let

\[
Y(\omega', z) = \sum_{n \in \mathbb{Z}} L(n)' z^{-n-2}.
\]

Then

\[
[L(n) - L(n)', u_m] = 0
\]

for \( m, n \in \mathbb{Z} \) and \( u \in V \). So \( L(-2) - L(-2)' \) acts as a constant on \( V \) as \( V \) is a simple vertex operator algebra. As a result, \( L(-2) - L(-2)' = 0 \) since the left side is both a constant and an operator which shifts degree by 2. The creation axiom for VOAs implies that, \( \omega' = \omega \). Since the central charge of \( \omega \) is 2, the central charge of \( \omega' \) is also 2. This implies that \( k = 4 \) and \( V \cong L(4, 0) \). Now \( V_L^+ \) is a \( L(4, 0) \)-module and the quotient module \( V_L^+/V \) has minimal weight (as inherited from \( V_L^+ \)) greater than 1. On the other hand, the minimal weight of the irreducible \( L(4, 0) \)-module \( L(4, \frac{i}{2} \alpha) \) is \( \frac{i(i+2)}{4(4k+2)} \), which is less than 2 for \( 0 \leq i \leq 4 \). Since every irreducible is one of these, we conclude \( V_L^+ = V = L(4, 0) \). Since \( V_L^- \) is an irreducible \( V_L^+ \)-module with minimal weight 1, we immediately see that \( V_L^- = L(4, 2\alpha) \).

So we have proved the following:

**Proposition 8.1.** If \( \text{rank}(L_1) = 2 \) there are two cases.

1. If \( L = L_{A_1^q} \) then \( V_L^+ \) is again a lattice vertex operator algebra \( V_K \) where \( K \) is generated by \( \beta_1, \beta_2 \) with \( (\beta_i, \beta_2) = 4 \) and \( (\beta_1, \beta_2) = 0 \). The automorphism group of \( V_L^+ \) is the LVOA* group which is isomorphic to the LVOA-group for lattice \( K \).
(2) If $L = L_{A_2}$, then $V_L^+$ is isomorphic to the vertex operator algebra $L(4,0)$ and $\text{Aut}(V_L^+)$ is isomorphic to $\text{PSL}(2,\mathbb{C})$ which is the automorphism group of $\text{sl}(2,\mathbb{C})$.

8.2 $\text{rank}(L_1) = 1$

8.2.1 $L$ rectangular.

We first assume that $L = \mathbb{Z}r + \mathbb{Z}s$ such that $(r,r) = 2$, $(s,s) \in 6 + 8\mathbb{Z}$ and $(r,s) = 0$. Then $V_L = V_{L_{A_1}} \otimes V_{Zs}$ and

$$V_L^+ = V_{L_{A_1}}^+ \otimes V_{Zs}^+ \oplus V_{L_{A_1}}^- \otimes V_{Zs}^-.$$

**Lemma 8.2.** A group of shape $(\mathbb{C}\beta/\mathbb{Z}_{1\beta}^+ \cdot \mathbb{Z}_2) \times \mathbb{Z}_2$ acts on $V_L^+$ as automorphisms.

**Proof.** We have already mentioned that $V_{L_{A_1}}^+$ is isomorphic to $V_{\mathbb{Z}_2\beta}$ for $(\beta, \beta) = 8$ and $V_{L_{A_1}}^-$ is isomorphic to $V_{\mathbb{Z}_2\beta + \frac{1}{2}\beta}^+$ as $V_{L_{A_1}}^+$-modules. We also know from [DN1] that $\text{Aut}(V_{\mathbb{Z}_2\beta})$ is isomorphic to $\mathbb{C}\beta/(\mathbb{Z}_{1\beta}^+ \cdot \mathbb{Z}_2)$ where the generator of $\mathbb{Z}_2$ is induced from the $-1$ isometry of the lattice $\mathbb{Z}_2\beta$. The action of $\lambda\beta \in \mathbb{C}\beta$ is given by the operator $e^{2\pi i\lambda\beta(0)}$. Note that $\mathbb{C}\beta$ acts on $V_{\mathbb{Z}_2\beta + \frac{1}{2}\beta}$ in the same way. But the kernel of the action of $\mathbb{C}\beta$ on $V_{\mathbb{Z}_2\beta + \frac{1}{2}\beta}$ is $\mathbb{Z}_{1\beta}^+ \beta$ instead of $\mathbb{Z}_{1\beta} \beta$. As a result, the torus $\mathbb{C}\beta/\mathbb{Z}_{1\beta}^+ \cdot \mathbb{Z}_2$ acts on both $V_{\mathbb{Z}_2\beta}$ and $V_{\mathbb{Z}_2\beta + \frac{1}{2}\beta}$. By [DG], $\text{Aut}(V_{\mathbb{Z}_2\beta}^+)$ is isomorphic to $\frac{1}{2}\mathbb{Z}_s/\mathbb{Z}_s \cong \mathbb{Z}_2$ which also acts on $V_{\mathbb{Z}_2\beta}^-$. So the group $(\mathbb{C}\beta/\mathbb{Z}_{1\beta}^+ \cdot \mathbb{Z}_2) \times \mathbb{Z}_2$ acts on $V_L^+$ as automorphisms. □

In order to determine $\text{Aut}(V_L^+)$ in this case we need to recall the notion of commutant from [FZ].

**Definition 8.3.** Let $V = (V,Y,1,\omega)$ be a vertex operator algebra and $U = (U,Y,1,\omega')$ be vertex operator subalgebra with a different Virasoro vector $\omega'$. The commutant $U^c$ of $U$ in $V$ is defined by

$$U^c := \{ v \in V | u_nv = 0, u \in U, n \geq 0 \}.$$

**Remark 8.4.** The above space $U^c$ is the space of vacuum-like vectors for $U$ (see [L1]).

**Lemma 8.5.** Let $V$ be a vertex operator algebra and $U^i = (U^i,Y,1,\omega^i)$ are simple vertex operator subalgebras of $V$ with Virasoro vector $\omega^i$ for $i = 1,2$ such that $\omega = \omega^1 + \omega^2$. We assume that $V$ has a decomposition

$$V \cong \bigoplus_{i=0}^p P^i \otimes Q^i,$$

as $U^1 \otimes U^2$-module such that $P^0 \cong U^1$, $Q^0 \cong U^2$, the $P^i$ are inequivalent $U^1$-modules and the $Q^i$ are inequivalent $U^2$-modules. Then $(U^1)^c = U^2$ and $(U^2)^c = U^1$. 

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Proof. It is enough to prove that \((U^2)^c \subset U^1\). Let \(v \in (U^2)^c\). Then \(v\) is a vacuum-like vector for \(U^2\). Then the \(U^2\)-submodule generated by \(v\) is isomorphic to \(U^2\) (see [L1]). Since \(V\) is a completely reducible \(U^2\)-module and any \(U^2\)-submodule isomorphic to \(U^2\) is contained in \(U^1 \otimes U^2\). In particular, \(v \in U^1 \otimes U^2\). This forces \(v \in U^1\). \(\square\)

Proposition 8.6. The group \(\text{Aut}(V^+_L)\) is isomorphic to \(((\mathbb{C}\beta/\mathbb{Z}^{1/4}_{\mathbb{R}}\beta) \cdot \mathbb{Z}_2) \times \mathbb{Z}_2\). This can be interpreted as an action of \(N(\hat{\mathbb{Z}}\beta) \times \mathbb{Z}_2\), where \((\beta, \beta) = 8\).

Proof. We have already shown 8.2 that the group \(((\mathbb{C}\beta/\mathbb{Z}^{1/4}_{\mathbb{R}}\beta) \cdot \mathbb{Z}_2) \times \mathbb{Z}_2\) acts on \(V^+_L\) as automorphisms.

Let \(\sigma\) be an automorphism of \(V^+_L\). Then \(\sigma\beta(-1) = \lambda\beta(-1)\) for some nonzero \(\lambda \in \mathbb{C}\) as \((V^+_L)_{1}\) is spanned by \(\beta(-1)\). This implies that \(\sigma\beta(n)\sigma^{-1} = \lambda\beta(n)\) for \(n \in \mathbb{Z}\).

Since \(V^+_z\) is precisely the subspace of \(V^+_L\) consisting of vectors killed by \(\beta(n)\) for \(n \geq 0\), we see that \(\sigma V^+_z \subset V^+_z\). Thus \(\sigma|_{V^+_z}\) is an automorphism of \(V^+_z\). On the other hand, \(V^+_{L_A}\) is the commutant of \(V^+_z\) in \(V^+_L\) by Lemma 8.5.

The above show that \(\sigma\) induces an automorphism of the tensor factor \(V^+_{L_A}\). The restriction of \(\sigma\) to \(V^+_L \otimes V^+_z\) is a product \(\sigma_1 \otimes \sigma_2\) for some \(\sigma_1 \in \text{Aut}(V^+_{L_A})\) and \(\sigma_2 \in \text{Aut}(V^+_z)\). Multiplying \(\sigma\) by \(\sigma_2\) we can assume that \(\sigma_2 = 1\) on \(V^+_z\). As we have already mentioned, \(\text{Aut}(V^+_{L_A})\) is isomorphic to \(((\mathbb{C}\beta/\mathbb{Z}^{1/4}_{\mathbb{R}}\beta) \cdot \mathbb{Z}_2)\). Since \((\mathbb{C}\beta/\mathbb{Z}^{1/4}_{\mathbb{R}}\beta)\) acts trivially on \(\beta(-1)\) and the outer factor \(\mathbb{Z}_2\) is represented in \(\text{Aut}(V^+_{L_A})\) by action of \(\pm 1\) on \(\beta(-1)\). As a result \(\sigma(-1) = \pm \beta(-1)\).

Now multiplying \(\sigma\) by an outer element of \(((\mathbb{C}\beta/\mathbb{Z}^{1/4}_{\mathbb{R}}\beta) \cdot \mathbb{Z}_2)\), we can assume that \(\sigma(-1) = \beta(-1)\).

Set \(W_{n\beta} = M(1) \otimes e^{n\beta} \otimes V^+_z\) and \(W_{n\beta + \beta/2} = M(1) \otimes e^{n\beta + \beta/2} \otimes V^-_z\) for \(n \in \mathbb{Z}\) where \(M(1) = \mathbb{C}[(\beta(-n)\vert n > 0)]\). Then \(V^+_L = \oplus_{n \in \mathbb{Z}} W_{n\beta}\) and \(u_m v = W_{\mu + \nu}\) for \(u \in W_{\mu}\) and \(v \in W_{\nu}\), and \(n \in \mathbb{Z}\). Note that \(W_{\mu}\) is the eigenspace of \(\beta(0)\) with eigenvalue \((\beta, \mu)\). Since \(\sigma(-1) = \beta(-1)\) we see that \(\sigma\) acts on each \(W_{\mu}\) as a constant \(\lambda_{\mu}\) and \(\lambda_{\mu} e_{\nu} = \delta_{\mu \nu}\). As a result, \(\sigma = e^{2\pi i \rho(0)}\) for some \(\gamma \in \mathbb{C}\beta\). That is, \(\sigma\) lies in \(\mathbb{C}\beta/\mathbb{Z}^{1/4}_{\mathbb{R}}\beta\).

This completes the proof. \(\square\)

8.2.2 \(L\) not rectangular

Next we assume that \(L \neq \mathbb{Z}r \perp \mathbb{Z}s\). Then \(L = \mathbb{Z}r \oplus \mathbb{Z}^{1/4}_{\mathbb{Z}}(s + t)\) where \((s, s) \in 6 + 8\mathbb{Z}\) and \((s, s) \geq 14\) (see 4.3). Let \(K = \mathbb{Z}r \oplus \mathbb{Z}s\). Then \(L = K \cup (K + \frac{1}{2}(r + s))\) and \(V_L = V_{Zr} \otimes V_{Zs} \oplus V_{(Z+\frac{1}{2})r} \otimes V_{(Z+\frac{1}{2})s}\). Thus

\[
V^+_L = V^+_{Zr} \otimes V^+_{Zs} \oplus V^-_{Zr} \otimes V^+_{Zs} \oplus V^+_L \otimes V^-_{K} \oplus V^+_{K} \otimes V^-_{Zr} \otimes V^+_{(Z+\frac{1}{2})s}
\]

and

\[
V^+_L = V^+_K \oplus V^+_K \otimes \mathbb{Z}^{1/4}_{\mathbb{Z}}(s + t).
\]
As before, we note that $V_{Zr}^+$ is isomorphic to $V_{Z\beta}$ with $(\beta, \beta) = 8$.

**Proposition 8.7.** Assume that $\text{rank}(L_1) = 1$, $L \neq Zr + Zs$, $r, s$ as above. Then $\text{Aut}(V_L^+) \cong (\mathbb{C}/\mathbb{Z}(\beta) \cdot \mathbb{Z}_2$, where $(\beta, \beta) = 8$. The action is trivial on the subVOA $V_{Zs}^+$ and leaves $V_{Zr}^+$ invariant. A generator of the quotient $\mathbb{Z}_2$ comes from the $-1$ symmetry of $\beta/\mathbb{Z}(\beta)$ and $\alpha \in \mathbb{C}/\mathbb{Z}$ acts as $e^{2\pi i \alpha(0)}$.

**Proof.** Note that $V_K^+$ is a subalgebra of $V_L^+$ and $V_{K+\frac{1}{2}(r+s)}^+$ is an irreducible $V_K^+$-module. By Proposition 8.6,

\[ \text{Aut}(V_K^+) = ((\mathbb{C}/\mathbb{Z}(\beta) \cdot \mathbb{Z}_2) \times \mathbb{Z}_2. \]

As we have already mentioned that $V_{Zr}^+$ is isomorphic to $V_{Z\beta}$ with $(\beta, \beta) = 8$ and $V_{Zr}^+$ is isomorphic to $V_{Z\beta+\frac{1}{2}r}$ as $V_{Z\beta}$-module. It is easy to see that $V_{(\beta+\frac{1}{2}r)} \cong \mathbb{Z}_2$ cannot be extended to an action of $V_L^+$. But the torus $\mathbb{C}/\mathbb{Z}(\beta)$ does acts on $V_L^+$. As a result, $N(\mathbb{Z}(\beta)) \cong (\mathbb{C}/\mathbb{Z}(\beta) \cdot \mathbb{Z}_2$ is a subgroup of $\text{Aut}(V_L^+)$.

The same argument used in the proof of Proposition 8.6 shows that any automorphism $\sigma$ of $V_L^+$ preserves $V_{Zr}^+ \otimes V_{Zs}^+$. Since $V_{Zr}^+ \otimes V_{Zs}^+$, $V_{Zr}^+ \otimes V_{Zs}^+$, $V_{(\beta+\frac{1}{2}r)} \otimes V_{(\beta+\frac{1}{2}s)}$, $V_{(\beta+\frac{1}{2}r)} \otimes V_{(\beta+\frac{1}{2}s)}$ are inequivalent irreducible $V_{Zr}^+ \otimes V_{Zs}^+$-modules (see [DM1] and [DLM]), we see that $\sigma$ preserves

\[ V_K^+ = V_{Zr}^+ \otimes V_{Zs}^+ \oplus V_{Zr}^+ \otimes V_{Zs}^+. \]

Since $\mathbb{C}/\mathbb{Z}(\beta) \cdot \mathbb{Z}_2$ is a quotient group of $\mathbb{C}/\mathbb{Z}(\beta) \cdot \mathbb{Z}_2$, we can multiply $\sigma$ by an element of $\mathbb{C}/\mathbb{Z}(\beta) \cdot \mathbb{Z}_2$ and assume that $\sigma$ acts trivially on the first tensor factor of $V_K^+$. If $\sigma$ is the identity on $V_K^+$, then $\sigma$ is either 1 or $-1$ on $V_{K+\frac{1}{2}(r+s)}^+$. If $\sigma$ is $-1$ on $V_{K+\frac{1}{2}(r+s)}^+$, then $\sigma = e^{\pi i \beta(0)}$ is an element of $\mathbb{C}/\mathbb{Z}(\beta) \cdot \mathbb{Z}_2$.

If $\sigma$ is not identity on $V_K^+$, then we must have $\sigma = e^{\pi i \frac{1}{2} \alpha(0)}$ on $V_K^+$. We will get a contradiction in this case. Notice that the lowest weight space of $V_{(\beta+\frac{1}{2}r)} \otimes V_{(\beta+\frac{1}{2}s)}$ is 1-dimensional and spanned by $u = (e^{r/2} + e^{-r/2}) \otimes (e^{s/2} + e^{-s/2})$. Since $\sigma$ preserves $V_{(\beta+\frac{1}{2}r)} \otimes V_{(\beta+\frac{1}{2}s)}$, it must map $u$ to $\lambda u$ for some nonzero constant $\lambda$. Note that $\lambda = \pm 1$. On the other hand,

\[ u_{-\frac{1}{4}(r+s)} = (e^r + e^{-r}) \otimes (e^s + e^{-s}) + \cdots \]

has nontrivial projection to the $-1$ eigenspace of $\sigma$ in $V_K^+$. This forces $\lambda = \pm i$, a contradiction. □
9 Appendix: Algebraic rules

For the symmetric matrices of degree \( n \), there is a widely used basis, Jordan product and inner product, which we review here. (This section is taken almost verbatim from [G10]).

**Proposition 9.1.** \( H \) is a vector space of finite dimension \( n \) with nondegenerate symmetric bilinear form \((\cdot, \cdot)\).

\( r, s, \ldots \) stand for elements of \( H \) and \( rs \) stands for the symmetric tensor \( r \otimes s + s \otimes r \).

\[
r s \times pq = (r, p)sq + (r, q)sp + (s, p)rq + (s, q)rp.
\]

\[
(rs, pq) = (r, p)(s, q) + (r, q)(s, p)
\]

**Definition 9.2.** The Symmetric Bilinear Form. Source: [FLM], p.217. This form is associative with respect to the product (Section 3). We write \( H \) for \( H_1 \). The set of all \( g^2 \) and \( x^+_\alpha \) spans \( V_2 \).

\[
\langle g^2, h^2 \rangle = 2\langle g, h \rangle^2,
\]

whence

\[
\langle pq, rs \rangle = \langle p, r \rangle\langle q, s \rangle + \langle p, s \rangle\langle q, r \rangle, \text{ for } p, q, r, s \in H.
\]

\[
\langle x^+_\alpha, x^+_\beta \rangle = \begin{cases} 2 & \alpha = \pm \beta \\ 0 & \text{else} \end{cases}
\]

\[
\langle g^2, x^+_\beta \rangle = 0.
\]

**Definition 9.3.** In addition, we have the distinguished Virasoro element \( \omega \) and identity \( \mathbb{I} := \frac{1}{2} \omega \) on \( V_2 \) (see Section 3). If \( h_i \) is a basis for \( H \) and \( h^*_i \) the dual basis, then \( \omega = \frac{1}{2} \sum_i h_i h^*_i \).

**Remark 9.4.**

\[
\langle g^2, \omega \rangle = \langle g, g \rangle
\]

\[
\langle g^2, \mathbb{I} \rangle = \frac{1}{2} \langle g, g \rangle
\]
\[ \langle \mathbb{I}, \mathbb{I} \rangle = \text{dim}(H)/8 \]

\[ \langle \omega, \omega \rangle = \text{dim}(H)/2 \]

If \( \{x_i \mid i = 1, \ldots, \ell \} \) is an ON basis,

\[ \mathbb{I} = \frac{1}{4} \sum_{i=0}^{\ell} x_i^2 \]

\[ \omega = \frac{1}{2} \sum_{i=0}^{\ell} x_i^2. \]

**Definition 9.5.** The product on \( V_2^F \) comes from the vertex operations. We give it on standard basis vectors, namely \( xy \in S^2H_1 \), for \( x, y \in H_1 \) and \( v_\lambda := e^\lambda + e^{-\lambda} \), for \( \lambda \in L_2 \). (This is the same as \( x_{\lambda}^+ \), used in [FLM].) Note that (3.1.1) give the Jordan algebra structure on \( S^2H_1 \), identified with the space of symmetric \( 8 \times 8 \) matrices, and with \( \langle x, y \rangle = \frac{1}{8} \text{tr}(xy) \). The function \( \varepsilon \) below is a standard part of notation for lattice VOAs.

\[ \begin{aligned}
(3.1.1) & \quad x^2 \times y^2 = 4\langle x, y \rangle xy, \\
& \quad pq \times y^2 = 2\langle p, y \rangle qy + 2\langle q, y \rangle py, \\
& \quad pq \times rs = \langle p, r \rangle qs + \langle p, s \rangle qr + \langle q, r \rangle ps + \langle q, s \rangle pr;
\end{aligned} \]

\[ \begin{aligned}
(3.1.2) & \quad x^2 \times v_\lambda = \langle x, \lambda \rangle^2 v_\lambda, \\
& \quad xy \times v_\lambda = \langle x, \lambda \rangle \langle y, \lambda \rangle v_\lambda \\
(3.1.3) & \quad v_\lambda \times v_\mu = \begin{cases} 0 & \langle \lambda, \mu \rangle \in \{0, \pm 1, \pm 3\}; \\
\varepsilon \langle \lambda, \mu \rangle v_{\lambda+\mu} & \langle \lambda, \mu \rangle = -2; \\
\lambda^2 & \lambda = \mu. \end{cases}
\end{aligned} \]

Some consequences are these:

**Corollary 9.6.** If \( x_1, \ldots \) is a basis and \( y_1, \ldots \) is the dual basis, then \( \mathbb{I} := \frac{1}{4} \sum_{i=1}^{\ell} x_i y_i \) is the identity of the algebra \( S^2H \).

\[ \langle \mathbb{I}, \mathbb{I} \rangle = \frac{\ell}{8}. \]
References


