Classification of finite quasisimple groups which embed in exceptional algebraic groups.

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Abstract. Since the early 80s, there have been efforts to determine which finite simple groups have a projective embedding into an exceptional complex algebraic group, i.e., one of $G_2(\mathbb{C})$, $F_4(\mathbb{C})$, $3E_6(\mathbb{C})$, $2E_7(\mathbb{C})$, $E_8(\mathbb{C})$ (note that these are simply connected groups). In this article, *we settle exactly which finite quasisimple groups embed in each exceptional algebraic group*, a result which is stronger than the title claim of the recent survey [GRS]. That is, we classify *QE-pairs*, all (Q, E) where Q is a finite quasisimple group, E an exceptional algebraic group such that there exists an embedding of Q in E. This requires a study of *all central extensions* of finite simple groups which have some projective representation in an exceptional group. We review proofs in the literature, make simplifications and fill certain gaps. We also give new details about many of the particular embeddings, such as conjugacy, characters on modules and Lie primitivity.

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1. Introduction.

This article resolves a number of questions about embeddings of finite quasisimple groups into exceptional algebraic groups, and achieves another milestone for the classification. *We settle exactly which finite quasisimple groups embed in each exceptional algebraic group in characteristic zero.*

(1.1) Definition. A *QE pair* is a pair (S, G), where G is an exceptional algebraic group and S is a finite quasisimple group for which there exists an embedding of S in G.

Main Theorem. All QE-pairs are known and are listed in Table QE.

The recent survey [GRS] reported on the study of finite subgroups of complex algebraic groups by many researchers, and gives the context for the present article. A case of particular interest is that of *finite quasisimple subgroups of the exceptional algebraic groups,* $G_2(\mathbb{C})$*,* $F_4(\mathbb{C})$*,* $3E_6(\mathbb{C})$*,* $2E_7(\mathbb{C})$ *and* $E_8(\mathbb{C})$. In [GRS], we described all pairs (F, G), where F is a finite simple group, G is an exceptional algebraic group for which there exists a *projective embedding* of F into G. A summary of what was known around 1998 about such pairs was given in Table PE [GRS]. Table QE supersedes Table PE. In particular, note that all question marks of Table PE have been removed. As before, the general problem of settling embeddings up to conjugacy remains open, though some new results are reported here.

Here is an overview of the sections in this article. In Section 1.1, we discuss the different meanings of "knowing an embedding" which occur in the literature on finite subgroups of exceptional groups. In Section 2, we present Table QE. In Section 3, we give a new proof of the completeness of the list of finite simple groups which have a projective embedding in $E_8(\mathbb{C})$. The finite simple groups in the second column of Table QE are exactly those not eliminated in Section 3. Finally, in Section 4, we give detailed justifications of the individual entries of Table QE. Some justifications follow from existing literature. A Table QE entry has a * if Section 4 contains some new result. Also, we tell what is known about fusion patterns, characters and primitivity.

A longer preprint version of [GRS], which contained all proofs necessary to justify Table PE, was circulated in spring, 1998. Since then, a shortened version of our preprint (without the proofs) was published as [GRS] and we intended to publish the omitted proofs in a form like the present article. Also since then, a paper of Liebeck and Seitz [LS] appeared which addresses the question of projective embedding of finite quasisimple groups in exceptional algebraic groups for the more general case of arbitrary characteristic. Our goal for the present article is to determine, in characteristic 0, the precise list of QE-pairs, not just settle existence of projective embeddings, and whenever possible, determine the embeddings of the first member of a QE-pair up to conjugacy. The proofs here do overlap the arguments of [LS] somewhat. For every finite simple group which is not eliminated in Section 3, we need to consider which perfect central extensions S are members of a QE-pair (S, G). This is routine for most F, but one encounters some nontrivial obstructions. Dealing with them is the subject of many special articles, surveyed in [GRS] (at least the pre-1999 ones).

For example, even if one knows existence of a particular projective embedding of F into $E_8(\mathbb{C})$, there may be no simple way to decide which other central extensions S of F embed into $E_8(\mathbb{C})$. There are many examples of this is in Section 4. For variety of central extensions, we mention the cases $F \cong PSL(3,4)$ and $PSU(4,3)$. For $PSL(2,27)$, existence of an embedding in $F_4(\mathbb{C})$ was proved in [CW97], but the problem of an embedding of $SL(2,27)$ into $E_8(\mathbb{C})$ resisted solution for years, and represented the last Table PE question mark to go. In a separate article [GR27], we prove that there are exactly twelve conjugacy classes of embeddings in $E_8(\mathbb{C})$, all of which factor through an embedding into a natural $2E_7(\mathbb{C})$ subgroup.

In characteristic 0, the set of QE-pairs is now known, and there is substantial work done on the conjugacy problem. In positive characteristic, the situation is much less definitive. Any finite simple group representing a row of Table QE has a projective embedding in arbitrary characteristic. A finite simple group may have no projective embeddings in characteristic 0, but have some in characteristic $p > 0$ if p divides the group order [GRS][LS]. Furthermore, in positive characteristic, techniques to classify embeddings up to conjugacy are less developed than in characteristic 0. For these two reasons, it would be hard to estimate when an arbitrary-characteristic analogue of Table QE might be attained. For characteristic $p > 0$ not dividing the order of the finite group S, the equivalence classes of embeddings of S into the same type of algebraic group in characteristics 0 and p are in bijection, by Larsen's Theorem, [GR31], Appendix. This means that many of the "hard cases" in characteristic 0 can be solved by computer work over integers modulo suitable p. See [GRS].

For an overview of the computing issues involved, see [GRS] and [GR27].

Our notations and definitions are fairly standard, but we list a few for clarity.

$W(\Phi)$ denotes the Weyl group of a root system of type Φ;

G^0 means the connected component of the identity in the algebraic group G;

$Emb(F, G)$, $ProjEmb(F, G)$ are the sets of embeddings, projective embeddings, resp., of the finite group F in the group G;

$Emb(F, G)/G$, $ProjEmb(F, G)/G$ denote the orbits of sets of embeddings under conjugacy by G.

Given groups S and G, a *fusion pattern* is a function f from conjugacy classes of S to conjugacy classes of G such that f preserves element order and respects the power maps.

As usual, $H.K$ denotes a group with normal subgroup H and quotient K. We make make some special extensions of this convention. The nonsplit central extension of $E_7(\mathbb{C})$ by a group of order 2 is referred to so often that we write $2E_7(\mathbb{C})$, without a dot between the 2 and $E_7(\mathbb{C})$. Similarly for $3E_6(\mathbb{C})$. Also, when the context indicates that finite quasisimple groups are under consideration, a notation $6Alt_6$ refers to a perfect group of the form $6{\cdot}Alt_6$. It would be inadequate to write $6{\cdot}Alt_6$ since such a group could be the nonperfect $2 \times 3{\cdot}Alt_6$ or $3 \times 2{\cdot}Alt_6$. Additional notations are used as special need arises.

Finally, we list some special abbreviations and where to find their definitions.

QE-pair (1.1) Quasisimple embedding pair;

AFPF (3.4) adjoint fixed point free action of a subgroup of the adjoint group;

RCC (3.5) reduction to the classical case;

XR (3.2) character restriction technique;

EFO (1.1) elements of finite order, or theory of such;

ZDC (Section 2) zero dimensional centralizer (this term is applied to a finite subgroup of an algebraic group which has zero fixed points on the adjoint module);

tame (4.10).

1.1. What does it mean to know $Hom(F,G)/G$?

Given a finite group F and an algebraic group G, we would like to understand the finite set $Hom(F,G)/G$ as well as possible. For G a general linear group, $GL(n,\mathbb{C})$, this set is in bijection with the set of (reducible) characters of F that have degree n. Even if one knows the character table, one does not automatically have corresponding homomorphisms.

From now on in this section, G is an *exceptional algebraic group.* We review what it means in practice to "know" $Hom(F,G)$ up to conjugacy. The standards represented in the literature do vary. Typically, it is either trivial or hard to show that $Hom(F,G)$ is nontrivial.

One can know explicit matrix representations for elements of $Hom(F,G)$ on some G-module, or know only fusion patterns, or (even less) only character values for elements of F on some G-module. In certain cases, one knows the fusion patterns which occur for any element of $Hom(F,G)$ but not the number of elements of $Hom(F,G)$ associated to each fusion pattern. There are cases of knowing $|Hom(F,G)|$ in which only partial fusion patterns have been worked out for each member of $Hom(F,G)$. See remarks below for a few examples which illustrates these points.

In the case of several F which have only Lie primitive embeddings, $|Hom(F,G)/G|$ was determined by studying only some of the conjugacy classes of F or by determining an explicit list of feasible characters and showing that each determines a single class of embeddings in $Hom(F,G)/G$. For an embedding of $PSL(2,q)$, for instance, the articles [CGL][GR31][GR41][GR27][S96] do not generally specify the $E_8(\mathbb{C})$-class of all elements of order dividing q^2-1. This would involve further work. Also, in the constructions of embeddings of $PSL(2,q)$ subgroups into $E_8(\mathbb{C})$, some elements of large order are *given* as elements of maximal tori, and so their membership in an $E_8(\mathbb{C})$ conjugacy class can be determined (in the sense of Section 1.2), though in practice this was not done, while other elements of large order were not given as elements of any maximal tori. We may know the spectra for such elements on the adjoint module, but this is in general insufficient to determine membership in a conjugacy class of $E_8(\mathbb{C})$. To illustrate, we mention that for $q=41$, elements of order 20 and 21, respectively, represent this dichotomy.

A final example illustrates several levels of explicitness. Many embeddings of $F \cong Alt_5$ in $G = E_8(\mathbb{C})$ factor through an embedding of a natural $HSpin(16,\mathbb{C})$ subgroup. This means that F has a degree 16 projective representation by orthogonal matrices, which can be described quite explicitly. To go from there to the representation on the E_8 adjoint module would be a matrix work project which has not been carried out, to our knowledge. In this example, the fusion pattern distinguishes the corresponding class in $Hom(F,G)/G$ unless we are in the unresolved ZDC case [Fr2][GRS].

1.2. EFO Theory.

Throughout this classification work we use the theory of elements of finite order in a connected algebraic group of characteristic 0. It has been summarized in many places. See [CG][GrElAb]. In particular, in a representation, an element of finite order is conjugate to the exponentiation of an element in the fundamental chamber of a Cartan subalgebra. A convenient way to label inner automorphisms of a fixed finite order is given by a labeling procedure for the extended Dynkin diagram [Kac]. This labeling also works for elements of order n in characteristic $p > 0$ if $(p, n) = 1$ [GrELAb].

For lists of elements of small finite orders in exceptional groups, see. e.g., [CG] [Fr1][Fr2][Fr3][Fr4][FrG] [GrElAb] [CW83][CW95][CW97]. The number of classes of elements of order n in a fixed quasisimple algebraic group grows rapidly with n, so displays of complete lists of elements of order n become undesirable. Instead, one can impose additional conditions on elements of order n and locate such elements by an exhaustive computer search of the labelings.

We do need to comment on a specialized point, that of deciding the order of an element of finite order if we know its order modulo the center. Let G be a simply connected quasisimple algebraic group. The *Ad-order* of an element is the smallest power which lies in the center, and this divides the order of such an element when the order is finite. The labeling procedure classifies elements of G of a given Ad-order, and further work with G-modules (or their weights in the weight lattice) is necessary to determine the orders of elements of G which represent a given element of order n in $G/Z(G)$.

2. Table of Embeddings of Finite Quasisimple Groups into Exceptional Algebraic Groups.

We now give Table QE which summarizes the state of the classification. It supercedes Table PE of [GRS]. Notation is like that of [CG], [GrElAb], which is derived from standard notation in finite group theory and the theory of Chevalley groups [Gor][Hup][Car].

A finite simple group L appears in column 2 of the table below if and only if it has a projective embedding in $E_8(\mathbb{C})$. If so, column 1 lists the isomorphism type z of the center of a quasisimple group S whose central quotient is L. In other words, each row of the table corresponds to a finite quasisimple group S.

For each quasisimple group S with a row in Table QE, we give what we know about embeddings in exceptional groups in columns 3,4,5,6,7; there is at least one embedding. In other language, *all QE-pairs are represented in Table QE, indicated by those entries which assert existence of an embedding. In short,* *we specify which central extensions of finite simple groups occur as subgroups in each of the five exceptional groups.* We do not, however, know a classification of all such embeddings, except for a minority of cases as indicated in the table. Note that since $E_8(\mathbb{C})$ contains all exceptional groups in simply connected versions via a chain in the order $G_2(\mathbb{C}) < F_4(\mathbb{C}) < 3E_6(\mathbb{C}) < 2E_7(\mathbb{C}) < E_8(\mathbb{C})$; a projective embedding into one of them is a projective embedding into $E_8(\mathbb{C})$. Note that the columns of Table QE are headed by simply connected groups.

An integer entry, m, means the quasisimple finite group S of that row has m equivalence classes of embeddings in the algebraic group H of that column, up to conjugacy in the algebraic group, that is $m = |Emb(S,H)/H|$. A bracketed entry (n) refers to the number $n = |Emb(S,H)/Aut(S) \times H|$. Entry y ("yes") means $|Emb(S,H)/H| \geq 1$.

An entry P (resp. p) means existence of an embedding and that all (resp. some) embeddings in that column's exceptional group are Lie primitive. An entry of an intermediate semisimple Lie group indicates existence of some (non Lie primitive) embedding via such a subgroup; in this case, there is no implication that all such embeddings occur this way. Z, S mean that there are unsettled cases only for zero dimensional centralizer (ZDC), small dimensional centralizer, respectively.

If a table entry has an asterisk *, further explanation is given in Section 4. (Generally, this means proof of something not already in the literature.) The column marked references cites sources for information in a given row. Exceptional isomorphisms are flagged by an arrow (so look to the right for an explanation). For two rows, $4PSL(3,4)$ and $6PSU(4,3)$, there is a possibility of two isomorphism types; see the last column and the text for further details.

See Section 4 for information on fusion patterns and characters.

The differences between Tables QE and PE are mostly refinements of information, but they do involve some errors in Table PE. The group $2{\cdot}Alt_{10}$ does embed in $F_4(\mathbb{C})$ and $3E_6(\mathbb{C})$ (see [CW97]). We are grateful to J-P Serre for informing us of this error. The group $2HJ$ does embed in $F_4(\mathbb{C})$ via its 6-dimensional symplectic representation. There are embedding of $2Alt_9$ in $2E_7(\mathbb{C})$ and Alt_{10} in $F_4(\mathbb{C})$. For several entries indicating a non-Lie primitive embedding, an inappropriate intermediate Lie subgroup was listed. See the rows for $2Alt_6, 2PSL(2,7), 2PSU(4,2)$ and M_{11}.

Table QE. The Finite Simple Groups with a Projective Embedding in $E_8(\mathbb{C})$.

z	Finite Simple Group	$G_2(\mathbb{C})$	$F_4(\mathbb{C})$	$3E_6(\mathbb{C})$	$2E_7(\mathbb{C})$	$E_8(\mathbb{C})$	Reference, Comments.
1	Alt_5	$4*$	13(8)	15(10)	19(12)	$\geq 31(\geq 19)Z$	$[Fr1, 2, 3, 4]$
2		$4*$	12(21)	18(32)	$\geq 96(\geq 51)$	$\geq 103(\geq 58)S$	$[Fr1, 2, 3, 4]$
1	Alt_6	0	$3A_2A_2$	y	y	y	$[CG][GrG2]$
2		0	$4A_3$	y	y	y	$[CG][GrG2]$
3		$2(1); 3A_2*$	$3A_2A_2$	y	y	y	$[CG][GrG2]$
6		0	$0*$	$2(1); 6A_5$	y	y	$[CG][GrG2]$
1	Alt_7	0	0	$6A_5$	2^2D_6	y	$[CG]$
2		0	$4A_3$	y	y	y	$[CG]$
3		0	$0*$	$6A_5$	y	y	$[CG]$
6		0	$0*$	$6A_5$	y	y	$[CG]$
	Alt_n						
1	$n = 8, \ldots, 10$	0	0	0	$n \leq 9$	$n \leq 10; 3A_8$	$[CG]$
2	$n = 8, \ldots, 17$	0	$n \leq 10$	$n \leq 11$	$n \leq 13$	$n \leq 17; 2D_8$	$[CG][CW95]$
	$PSL(2, 4) \rightarrow$						$\cong Alt_5$
	$PSL(2, 5) \rightarrow$						$\cong Alt_5$
1	$PSL(2, 7)$	$2*$	y	y	y	$\geq 33(\geq 16)S$	$[K]$
2		0	$4A_3$	y	y	$\geq 55(\geq 22)S$	$[K]$
1	$PSL(2, 8)$	$3(1)P$	y	y	y	y	$[CW83][GrG2]$
	$PSL(2, 9) \rightarrow$						$\cong Alt_6$
1	$PSL(2, 11)$	0	$0*$	$5A_4, 4D_5$	y	y	$[CG][CW83]$
2		0	$0*$	$4D_5, 6A_5$	y	y	$[CG][CW83]$
1	$PSL(2, 13)$	$2(1)P$	y	$\geq 6(\geq 3)$	y	y	$[CW83][GrG2]$
2		0	$2A_1C_3$	$2(1); 2A_1A_5*$	y	y	
1	$PSL(2, 16)$	0	$0*$	$0*$	$0*$	$2D_8$	$[CG][CW97]$
1	$PSL(2, 17)$	0	$2B_4, p$	y	$4A_7$	$3A_8$	$[CG]$
2		0	0	0	0	$2D_8$	$[CG]$
1	$PSL(2, 19)$	0	0	$\geq 4(1)P*$	y	A_8	$[CG][CW97]$
2		0	0	0	P	y	$[S96]$
1	$PSL(2, 25)$	0	P	y	y	y	$[CW97]$
2		0	0	0	$0*$	$2(1); 2D_8$	$[CG]$
1	$PSL(2, 27)$	0	$\geq 3(1)P$	y	y	y	$[CW97]$
2		0	0	0	$12(2)P$	12(2)	$[CW97][GR27]$
1	$PSL(2, 29)$	0	0	0	$0*$	$2B_7 \leq 2D_8$	$[CG], p370$
2		0	0	0	$P*$	$y*$	$[SP]$

Finite Simple Group	z	G_2	F_4	$3E_6$	$2E_7$	E_8	Reference, Comments.
1	$PSL(2,31)$	0	0	0	0	$\geq 3(2)P$	$[S96][GR31]$ 3(2) for $PGL(2,31)$
1	$PSL(2,32)$	0	0	0	0	$5(1)P$	$[GR31]$
2	$PSL(2,37)$	0	0	0	$2(1)P$	$2(1)$	$[KR][CG],(5.2.10)$
1	$PSL(2,41)$	0	0	0	0	$3(1)P$	$[GR41]$
1	$PSL(2,49)$	0	0	0	0	$2(1)P$	$[GR41]$
1	$PSL(2,61)$	0	0	0	0	$2(1)P*$	$[CGL]$
	$PSL(3,2) \rightarrow$						$\cong PSL(2,7)$
1	$PSL(3,3)$	0	P	y	y	y	$[Alek][CG][GrElAb]$; in $3^3{:}SL(3,3)$
2	$PSL(3,4)$	0	0	0	$4A_7$	y	$[CG]$
4		0	$0*$	$0*$	$0*$	$8A_7$	$[CG]$; embeds in $SL(8,\mathbb{C})$
6		0	0	$6A_5$	y	y	$[CG]$
1	$PSL(3,5)$	0	0	0	0	P	in $5^3{:}SL(3,5)$; $[Alek][CG][GrElAb]$
	$PSL(4,2) \rightarrow$						$\cong Alt_8$
1	$PSU(3,3)$	1	y	y	y	y	$[CW83][GrG2]$
1	$PSU(3,8)$	0	0	0	$1P*$	$1*$	$[GRU]$; $PSU(3,8){:}12 \leq 2E_7(\mathbb{C})$
1	$PSU(4,2)$	0	$0*$	$6A_5$	y	y	$[CG]$; $\cong \Omega^-(6,2) \cong W'_{E_6}$
2		0	$4A_3$	y	y	$2D_8$	
6	$PSU(4,3)$	0	0	$6A_5$	y	y	$[CG]$; embeds in $SL(6,\mathbb{C})$
1	$PSO^+(8,2)$	0	$0*$	$0*$	$y*$	$2^2D_4D_4$	$[CG]$
2	$PSO(8,2)$	0	$0*$	$0*$	$0*$	$2^2D_4D_4$	$[CG]$
2^2	$PSO(8,2)$	0	2^2D_4	y	y	$2^2D_4D_4$	$[CG]$
	$PSp(4,3) \rightarrow$						$\cong PSU(4,2)$
2	$PSp(4,5)$	0	0	0	$0*$	B_6	$[CG]$
1	$PSp(6,2)$	0	$0*$	$0*$	$y*$	$2^2D_4^2$	$[CG]$; $\cong \Omega(7,2) \cong W'_{E_7}$
2		0	2^2D_4	y	y	$2^2D_4^2$	$[CG]$
1	$Sz(8)$	0	0	0	0	$3(1)P$	$[GR8]$
	$G_2(2)' \rightarrow$						$\cong PSU(3,3)$
1	$G_2(3)$	0	0	0	$0*$	D_7	$[CG]$
1	${}^3D_4(2)$	0	P	y	y	y	$[CW97]$
1	${}^2F_4(2)'$	0	0	P	y	y	$[CW97]$; ${}^2F_4(2)$ embeds in $3E_6(\mathbb{C})$
1	M_{11}	0	0	$4D_5$	y	y	$[CG]$
2	M_{12}	0	0	$0*$	$2B_5 \leq 4D_6$	y	$[CG]$
2	HJ	0	$2A_1C_3$	$6A_5$	y	y	$[CG]$

3. Proof of the Classification of Finite Simple Groups which Projectively Embed in Algebraic Groups.

In this section, we justify our table of finite simple groups that projectively embed in $E_8(\mathbb{C})$. In Section 4, this analysis is refined to get exact central extensions which occur. Many of the arguments that we use are similar in detail to arguments of [CG] and the more recent [LS]. However, we try to apply recent developments to streamline our arguments wherever possible. Our determination of exact central extensions does involve significant character theory work beyond what is necessary to settle projective embeddings.

The following compendium of known results about subgroups of $G = E_8(\mathbb{C})$ is our primary tool. All Lie algebras are complex unless specified otherwise.

(3.1) Proposition. (a) (Borel-Serre, [BS]) A supersolvable subgroup of $Aut(\mathfrak{g})$, for a finite dimensional simple Lie algebra $\mathfrak{g}$, normalizes a maximal torus.

(b) Let H be an algebraic group. Suppose that E is contained in a torus T of H. Then $N_H(T)$ covers $N_H(E)/C_H(E)^\circ$ (that is, $N_H(E) \leq N_H(T)C_H(E)^\circ$).

(c) An element of the Weyl group of E_8 has order at most 30.

(d) Suppose that $E_8(\mathbb{C})$ contains Y, that Y is isomorphic to a Borel subgroup of $PSL(2,q)$ or $SL(2,q)$, that q is a power of the prime p and $O_p(Y)$ is toral. Then $q \leq 61$.

(e) A finite abelian subgroup of $E_8(\mathbb{C})$ with at most two generators is toral.

(f) If $E_8(\mathbb{C})$ has a finite non-abelian p-subgroup, then $p \leq 7$.

(g) Suppose that $E \cong p^n$ is a non-toral elementary abelian subgroup of $E_8(\mathbb{C})$. Then either $p = 5$ and $n = 3$, $p = 3$ and $n = 3$ or 4, or $p = 2$ and $3 \leq n \leq 9$. If in addition, E is maximal, then the possibilities for E up to conjugacy and the structure of $N_{E_8(\mathbb{C})}(E)$ are given in (1.8) [GrElAb]. When $p = 5$, $n = 3$. When $p = 3$, $n = 5$. When $p = 2$, $n = 8$ or 9.

(h) "$3B$ criterion". In $E_8(\mathbb{C})$, if an element x of order 3 satisfies $x \in C(x)'$, it is in class $3A$ or $3B$. If x is a commutator of two elements of order 3 in $C(x)$, then x is in $3B$.

Proof. [CG][GrElAb].

(3.2) Procedure XR. In this article, XR stands for "character restriction". Although we prefer to base our arguments on the group theoretic considerations of Proposition 3.1, we sometimes use character theoretic considerations. In particular, we sometimes show that there is no way to restrict the adjoint character of a quasisimple algebraic group H, to a particular finite group S. We use a simple computer program to locate any potential character restrictions that we require. The program works with partial characters of S: that is the restrictions of proper characters of S to some subset of conjugacy classes of S. Suppose that C is a set of classes in S and that χ is a partial character for S defined on C, we say that χ is an *H-feasible partial character* if, for each class $c \in C$, there is a class c' in H such that elements of c

and c' have the same order, and such that the value of χ at c equals the value of the adjoint character at c'. We shall be particularly interested in E_l-feasibility for $6 \leq l \leq 8$. We shall refer to our program as *The XR procedure from H to the finite group S using the set of classes C*. If the program shows that there is no feasible partial character, we can immediately conclude that S does not embed in H.

In order to implement a version of the XR-procedure for a group H with $H/Z(H) \cong E_l(\mathbb{C})$, we consider a collection $C = (c_1, c_2, \ldots, c_k)$ of conjugacy classes of S that includes the identity element as c_1. In our particular implementation, we further assume that all classes in C correspond to elements of order at most 4 when $l = 8$, and at most 3 when $l \in \{6, 7\}$. For each vector $X = (x_1, x_2, \ldots, x_k)$, where x_i is the value of the adjoint character of $E_l(\mathbb{C})$ at an element of the same order as elements of c_i, we use a backtrack search to determine whether X is a partial character on the set C. (Note that our restriction to elements of low order means that there are relatively few possibilities for the vector X, and these possibilities can be determined from the tables of low order elements in [CG] and [CW97].)

To accomplish the backtrack search to determine whether X represents a feasible character, we first order those irreducible characters of S, with degrees that do not exceed $dim(H)$, in decreasing degree order. We then use a standard greedy algorithm to construct all combinations of characters with degree less than or equal to $dim(H)$. When the degree is $dim(H)$, we compare the corresponding partial character to X: if they agree, we print out the corresponding combination of irreducible characters. We found that our simple program could complete its work in a matter of minutes even for cases with around 15 irreducible characters and sets of up to 5 conjugacy classes.

Exactly the same approach can be applied to obtain feasible restrictions of the 27-dimensional character of $3E_6(\mathbb{C})$ and the 56-dimensional character of $2E_7(\mathbb{C})$. We shall refer to the corresponding implementations as the XR-procedure for $3E_6$ and $2E_7$, respectively.

In many cases, we can see immediately that a particular group S does not embed in $E_8(\mathbb{C})$ because there is a set of classes C of S for which there are no E_8-feasible partial characters. However, we shall sometimes come across a group S, that we wish to rule out of consideration as a subgroup of $E_8(\mathbb{C})$ even though there appear to be E_8-feasible characters. The following Lemma is often applicable.

(3.3) Lemma. Suppose that S is a finite quasisimple group. Let C be a subset of the conjugacy classes of S.

(a) If S can be embedded in $E_8(\mathbb{C})$, then it is Lie primitive in a closed subgroup of one of the types $E_8, E_7, E_6, F_4, G_2, A_l, D_l$, B_m, C_m, where $1 \leq l \leq 8$, and $1 \leq m \leq 7$.

(b) If S has no projective representation with degree less than 10, then there is no projective embedding of S into a Lie group of type A_l with $l \leq 8$. Moreover there is no projective embedding of S into a Lie group of type G_2.

(c) If S has no projective representation with degree less than 17 that supports an invariant orthogonal form, then there is no projective embedding of S into a Lie group of type D_l with $l \leq 8$, or into a Lie group of type B_m with $1 \leq 7$.

(d) If S has no projective representation with degree less than 15 that supports an invariant symplectic form, then there is no projective embedding of S into a Lie group of type C_l with $l \leq 7$.

(e) If S has a Lie primitive embedding into $E_8(\mathbb{C})$, then S has an $E_8(\mathbb{C})$-feasible partial character on C that has a decomposition that does not include the trivial partial character.

(f) If S has a Lie primitive embedding into $2E_7(\mathbb{C})$, then S has a $2E_7(\mathbb{C})$-feasible partial character on C that has a decomposition that does not include the trivial partial character, and S has an $E_8(\mathbb{C})$-feasible partial character that has a decomposition that has exactly three copies of the trivial partial character.

(g) If S has a Lie primitive embedding into $F_4(\mathbb{C})$ or $3E_6(\mathbb{C})$, then S has a $3E_6(\mathbb{C})$-feasible partial character on C that has a decomposition that does not include the trivial partial character.

Proof. (3.3.g). This is clear if S has a Lie primitive embedding in $G \cong E_6(\mathbb{C})$, so suppose it has one in F, a natural $F_4(\mathbb{C})$ subgroup of G. We may assume that S has fixed points on the E_6-adjoint module, whence the fixed point subalgebra, $\mathfrak{a}$, is nonzero and reductive. So, S is in H, the centralizer in $E_6(\mathbb{C})$ of $\mathfrak{a}$, a proper subgroup. Since F has decomposition $52 + 26$ of the adjoint G-module, $H \not\geq F$ and so $S < H \cap F$, a proper, closed subgroup of F, contradicting Lie primitivity of S in F.

(3.4) Definition. We shall say that an embedding of a finite group into an algebraic group is *adjoint fixed point free* (AFPF) if the action of the finite group on the adjoint module is fixed point free.

A Lie primitive embedding of a finite group in a connected semisimple Lie group is AFPF.

(3.5) Procedure RCC. This stands for "reduction to the classical case", and it is a strategy for eliminating possible projective embeddings of a given finite simple group L.

We have $ProjEmb(L, E_8(\mathbb{C})) = \emptyset$ if (1) L has no AFPF embeddings in the adjoint groups $E_6(\mathbb{C})$, $E_7(\mathbb{C})$ and $E_8(\mathbb{C})$; and (2) L has no projective embeddings in any classical subgroup of $E_8(\mathbb{C})$, meaning any subgroup of type A_n and D_n for $n \leq 8$, or C_m and B_m, for $m \leq 7$.

Proof. Suppose that S is a perfect central extension of L contained in $E_8(\mathbb{C})$. By (1), S lies in a proper closed subgroup H of $E_8(\mathbb{C})$ which we may assume to be reductive and perfect and that S projects nontrivially to all the central factors of H. Choose such an H of least possible dimension. By (2), no central factor has classical type, so H is a central product of exceptional groups. If there are two or more factors, H has type G_2G_2 or G_2F_4, which violates (2) since $G_2(\mathbb{C})$ embeds in $B_3(\mathbb{C})$. So, H is quasisimple and exceptional. But then, (1) and minimality of $dim(H)$ are in contradiction. $\square$

The point is that projective embeddings of a finite simple group in any classical group are routine to settle with the character table (of the covering group of L), whereas we save work in an exceptional group if we need to deal only with embeddings of the simple group L and only on the adjoint module of that exceptional group. This step may be done, for example, with Procedure XR (3.2).

Exclusion from the list of rows in Table QE.

(3.6) Notation. We write G for the group $E_8(\mathbb{C})$, L for a finite simple group under consideration and S for a quasisimple group with quotient L. We assume that S embeds in $E_8(\mathbb{C})$. If L has Lie type, we write q for its defining field, p for its defining characteristic, B for a Borel subgroup of S and U for the unipotent part of B. This follows fairly standard practice, as in [Car][Gor][GrTwelve].

We now show that any finite simple group that is not listed in column two of Table QE has no projective embeddings into $E_8(\mathbb{C})$. We also give references for embeddings in the case of finite simple groups not eliminated, here or in Section 4, which treats actual table entries in greater detail.

Our program is to go through the list of finite simple groups, assumed to be complete. First we begin with the groups of Lie type, then move to the alternating and sporadic groups. A few points require [GrElAb].

Type A, rank 1. Let $L \cong A_1(p)$. The list of orders of elements in the Weyl group of E_8 and (3.1.d,e) show that $p \in \{2, 3, 5, 7, 11, 13, 17, 19, 29, 31, 37, 41, 61\}$. The first two values of p do not correspond to simple groups, while all remaining cases are represented by known embeddings. Later, we discuss embeddings of each of these in each of the exceptional groups.

If $L \cong A_1(p^2)$, (3.1.b,e) show that if p is odd, $(p^2-1)/2$ is the order of an element in $W(E_8)$. Thus, by (3.1.c), $p^2 \in \{4, 9, 25, 49\}$ and all these cases do lead to embeddings.

If $L \cong A_1(p^n)$, with $n \geq 3$, then (3.1.b,c) show that $p^n \in \{2^3, 2^4, 3^3\}$, or that U is not toral in $E_8(\mathbb{C})$. Hence (3.1.g) shows that either $p = 5$ and $n = 3$, or $p = 3$ and $n \in \{3, 4\}$, or $p = 2$ and $3 \leq n \leq 9$. Moreover, when $p = 2$, the Sylow 2-subgroup of L is pure and $n \leq 5$ [CG]. Thus $q = p^n \in \{2^3, 2^4, 2^5, 3^3, 3^4, 5^3\}$.

Case: $q = 125$. Here, U is not toral and so we we get an element of order 62 in $N(U)/U \cong SL(3, 5)$, a contradiction [GrElAb].

Case: $q = 81$. In this case, we use the results and notations of [GrElAb] (1.8) Table II. If U is toral, we get an element of order 40 in the Weyl group, contradiction. If U is not toral, it embeds as the $3B$-radical of E, a maximal nontoral elementary abelian 3-group; $|E| = 243$ and E has type 1, whence the fixed point subalgebra of U is a natural A_2 and E lies in a $3A_2E_6$ subgroup; U is contained in and E projects faithfully into the $3E_6$ factor. Since B induces on U a cyclic group of order 40, which is incompatible with the structure of the rank 4 type 1 $3E_6$-nontorals and their normalizers, we have a contradiction.

We conclude that the only possibilities are $L \in \{A_1(2^3), A_1(2^4), A_1(2^5), A_1(3^3)\}$. Examples of embeddings are known in all of these cases [CG][CW97][GR31].

Type A, rank at least 2. We next consider the case $L \cong A_2(p)$. Any projective embedding of L into $E_8(\mathbb{C})$ gives an embedding of $AGL(2,p)$ into $E_8(\mathbb{C})$. Hence, by (3.1.b,e), there is an element of order p^2-1 in W_{E_8}, thus $p^2-1 \leq 30$ and $p \in \{2,3,5\}$. Projective embeddings are known in all these cases.

Case: $L \cong A_2(p^n)$, with $n > 1$. Since L contains a central extension of $A_2(p)$, we need only consider embeddings in the cases $p \leq 5$. Now, S has a pure elementary abelian subgroup of order p^{2n} with normalizer of shape $AGL(2,p^n) \cong p^{2n}{:}GL(2,p^n)$. From [GrElAb], such a pure subgroup can exist only for $q = p^n \in \{2^2, 3^2\}$. Moreover, we can rule out $S \cong A_2(3^2)$ by Procedure XR because there is no E_8-feasible partial character of $A_2(3^2)$ on the classes 1A, 2A (four irreducible characters with degrees 1, 90, 91 and 91 must be considered). Many projective embeddings of $A_2(2^2)$ are known [CG].

Here is an alternative argument for the case $L \cong A_2(p^n)$, $p^n = 3^2$. There is a subgroup H of S of the form $AGL(2,9) \cong 3^4{:}GL(2,9)$. Let $A := O_3(H)$ and let K complement A in H. The 3B criterion shows that A is $3B$-pure and that the fixed point subalgebra, $\mathfrak{a}$, has dimension 8. We know that A is not toral (since a Singer cycle in $GL(2,9)$ has order 80), so we look up the classification of nontorals in [GrElAb], (1.8), Table III and see that A is not maximal nontoral, but is contained in a maximal one, E, of rank 5. Up to conjugacy, there are two possible E (called types 1 and 2), and both have a $3B$-pure hyperplane whose complement avoids $3B$; A must be this hyperplane. Now the reader should refer to Section 11 of [GrElAb], especially (11.5). Type 2 is impossible here since in that case A would be toral. So, E has type 1, and in this case A is not toral but has $\mathfrak{a}$ (see above) of type A_2 in a natural G_2. Let G_0 be the corresponding connected A_2 subgroup. Then H acts on G_0 with kernel containing A. So, A is a subgroup of $G_1 := C_G(G_0) \cong 3E_6$ and it is nontoral there since it is nontoral in G. The structure of $N_G(G_0) = 3A_2E_6{:}2$ shows that $N_{H\cap G_1}(A)$ induces at least $SL(2,9).4$ on it. But this is incompatible with the structure of the E_6-nontorals and their normalizers (see Table II, the case $Z=1$, $p=3$).

Case: $L \cong A_3(p^n)$. Since S has a subgroup of type $A_2(p^n)$, we need only consider the cases $q = p^n \in \{2,3,4,5\}$. An embedding is known for the case $q=2$.

In the case $q=5$, S has a subgroup $AGL(3,5) \cong 5^3{:}GL(3,5)$. The normal elementary abelian 5^3 must be toral (since the normalizer of a non-toral 5^3 is merely $5^3{:}SL(3,5)$ [GrElAb]), and now (3.1.b,c) rule out an embedding. If $q=4$, S has a pure elementary abelian subgroup of order 2^6, in contradiction to [CG]. Finally the case $q=3$ is ruled out because (1) there is no E_8-feasible partial character of the simple group

$A_3(3)$ on the classes 1A, 2A, 2B, 3A (seven irreducible characters with degrees 1, 26, 39, 52, 65, 90, and 234 are considered), and (2) there is no E_8-feasible partial character of the double cover of $A_3(3)$ on the classes of elements orders 1, 2 and 3 (nine irreducible characters with degrees 1, 26, 39, 40, 52, 65, 90, 208, and 234 are considered).

Case: $S \cong A_4(p^n)$. Here L has a subgroup $A_3(p^n)$, so we can only have $p^n = 2$. However, there is no E_8-feasible partial character of the group $A_4(2)$ on the classes 1A, 2A, 2B, 3A (five irreducible characters with degrees 1, 30, 124, 155, and 217 are considered). We conclude that there are no feasible projective embeddings of groups $L \cong A_l(p^m)$ for $l \geq 4$.

The unitary groups, twisted type A. Let $L \cong {}^2A_l(p^n)$, $l \geq 2$. Here, S has a non-abelian Sylow p-subgroup, so (3.1.f) shows that we need only consider unitary groups in characteristic $p \leq 7$.

We begin with groups $L \cong PSU(3,q)$. The smallest degree of a faithful character of the Borel subgroup of S is $(q-1)q$. Therefore, we need only consider values of q with $(q-1)q \leq 248$, i.e. $q \leq 16$. Since $p \leq 7$, $q = 16$ or $q \leq 9$. Note that $q \geq 3$ since $PSU(3,2)$ is solvable. There are embeddings of $PSU(3,3)$ in $E_8(\mathbb{C})$, e.g. via a B_3 subgroup. For embeddings of $PSU(3,8)$, see [GRU]. We shall demonstrate that there are no further feasible restrictions of the adjoint character of $E_8(\mathbb{C})$ to S, when $L \cong {}^2A_2(q) \cong PSU(3,q)$ for $q \geq 4$ and it suffices to treat $q \in \{4,5,7,9,16\}$.

Assume $L \cong PSU(3,4)$. The smallest faithful (projective) representation of L has degree 12, and the smallest faithful orthogonal (projective) representation of L has degree 24, so by (3.3.a,b,c,d) and Procedure RCC, any projective embedding of L into $E_8(\mathbb{C})$ comes from an AFPF embedding into $E_6(\mathbb{C})$, $E_7(\mathbb{C})$, or $E_8(\mathbb{C})$. To rule out these possibilities, we work with 8 irreducible characters with degrees 1, 12, 13, 39, 52, 64, 65, and 75 and consider partial characters on the set of classes 1A, 2A, and 3A. The only E_8-feasible partial characters have between 1 and 16 copies of the trivial partial character, so by (3.3.e) do not correspond to Lie primitive embeddings. Similarly, the only E_7-feasible partial characters involve at least 3 copies of the trivial partial character. Procedure XR for $3E_6$, using classes 1A and 2A of $PSU(3,4)$ shows that the only feasible restrictions of the 27-dimensional character of $3E_6(\mathbb{C})$ would have trace -5 on involutions of $PSU(3,4)$. Hence, we need only consider E_6-feasible partial characters that have trace 14 on involutions of $PSU(3,4)$. Procedure XR for E_6, applied to the classes 1A, 2A, and 3A of $PSU(3,4)$ now shows that all such E_6-feasible partial characters involve the trivial partial character with multiplicity at least 13. We conclude by (3.3.f,g) that there are no projective embeddings of L into $E_8(\mathbb{C})$.

We now eliminate $L \cong PSU(3,5)$. The group L has no faithful projective representation of degree less than 20. Moreover, an application of Procedure XR using the classes 1A, 2A, 3A shows that there is no E_6-feasible partial character without copies of the trivial partial character. We deduce from (3.3.a,b,c,d,g) and Procedure RCC that any projective embedding of L into $E_8(\mathbb{C})$ must arise from an AFPF embedding into $E_8(\mathbb{C})$ or $E_7(\mathbb{C})$. When we apply the Procedure XR to determine E_7-feasible characters of L, using the classes of elements of orders at most 3, we find two feasible partial characters that do not contain copies of the trivial partial character. These partial characters correspond to the irreducible character decompositions $21 + 28 + 84$, and $28 + 105$, where 21, 28, 84, and 105 represent any of the characters of the simple group with the corresponding degrees. In all corresponding combinations of characters, elements of order 5 have a rational trace, but not all elements of order 5 have trace 8. However, the only elements of order 5 in $E_7(\mathbb{C})$ that have a rational trace on the adjoint module have trace 8 [CG]. In particular, there is no AFPF embedding of L into $E_7(\mathbb{C})$. Similarly, when we apply the Procedure XR to determine E_8-feasible characters of L using the classes 1A, 2A, 3A, we find two feasible partial characters that do not contain copies of the trivial partial character. These two partial characters correspond to the irreducible character decompositions $20 + 84 + 144$, and $20 + 20 + 20 + 20 + 28 + 28 + 28 + 84$, where 20, 28, 84, and 144 represent any of the characters of the simple group with the corresponding degrees. The first of these character decompositions leads to a character value of $-1 - b7$ or its algebraic conjugate on elements of order 7, a contradiction since every semisimple element is real. The second decomposition leads to a character value of -4 on elements of order 7, but there is no element of $E_8(\mathbb{C})$ with this trace on the adjoint module [CG].

In the case $L \cong PSU(3, 7)$, there is no E_8-feasible partial character on the classes 1A, 2A, 3A (four irreducible characters with degrees 1, 42, 43, and 43 must be considered).

The case $q = 8$ is treated as follows. There is an embedding of $PSU(3, 8)$ [GRU] but we need to eliminate the case $S \cong SU(3, 8)$. Let z be an element of order 3 generating $Z(S)$. Then by the $3B$-criterion (3.1.h), z is in class $3B$ and so S embeds in the $3E_6$ factor of $C(z)$. But this factor has an irreducible module of dimension 27, whereas $(q-1)q \le 27$ fails for $q = 8$.

In the case $L \cong PSU(3, 9)$, Procedure XR shows that there are no E_8-feasible partial characters on the classes 1A, 2A, 3A, 3B (we need to use four irreducible characters of degrees 1, 72, 73 and 73 in this analysis).

We now eliminate the case $L \cong PSU(3, 16)$ by local analysis. We recall the inequality $q(q-1) \le 248$ used before. For $q = 16$, this gives $240 \le 248$, and means that B has a single faithful irreducible, which has dimension 240. The remaining 8 dimensions afford only nonfaithful irreducibles. Since $Z(U)$ acts by only nontrivial characters on the 240 degree irreducible, this 8-dimensional space must be its fixed point subalgebra; call it $\mathfrak{u}$. Since $Z(U)$ is pure, it must be 2B-pure and so this subalgebra is a Cartan subalgebra [CG], (3.8.iii). Consider the action of $B/Z(U)$ on $\mathfrak{u}$. The unique minimal normal subgroup is $U/Z(U)$. Since the centralizer in G of a Cartan subalgebra is a maximal torus, $U/Z(U)$ acts faithfully on $\mathfrak{u}$ since U is nonabelian and $U/Z(U)$ is minimal normal in B/Z. However, B acts transitively on $U/Z(U)$, which has order 2^6, and so faithful action on an 8 dimensional space is impossible.

For unitary groups, $PSU(n,q)$ with $n \geq 4$, projective embeddings are known for $PSU(4,2)$ and $PSU(4,3)$. We can rule out any other embeddings by disposing of the cases $L \cong PSU(4,8)$, $PSU(5,2)$, $PSU(5,3)$, since any other unitary group would have to contain an already eliminated unitary subgroup.

Let $L \cong PSU(4,8)$. Here, S contains a subgroup $SU(3,8)$ which does not embed in $E_8(\mathbb{C})$, as seen above, so we are done.

Let $L \cong PSU(5,3)$. We take a parabolic subgroup P of the form $P = RK$, where $R \cong 3^{1+6}$, $K \cong U(3,3)$. The action of U on R/R' is that of $U(3,3)$ on its natural module. By (3.1.h), $Z(R)$ is generated by an element of the class 3B, so P is contained in $H := C_G(Z(R))$, which has shape $3A_2E_6$. Since $PSU(3,3)$ does not projectively embed in $SL(3,\mathbb{C})$, P' lies in the $3E_6$-factor. We now have a projectively elementary abelian 3-group in $3E_6$. By the classification in [GrElAb], Table II, R must be embedded in a group of the form $3^{1+2} \times 3 \times 3$, a contradiction.

In the case $L \cong PSU(5,2)$, we work with 15 irreducible characters with degrees 1, 10, 11, 44, 55, 55, 66, 110, 110, 110, 120, 165, 176, 220, 220 on the set of classes 1A, 2A, 2B, 3E, and 3F. The only E_8-feasible partial characters include at least 6 copies of the trivial character. Thus by (3.3.f), there is no Lie primitive embedding of L into $E_8(\mathbb{C})$ or into $E_7(\mathbb{C})$. However, the smallest faithful projective representation of L has degree 10, and the smallest faithful orthogonal projective representation of L has degree 20. Hence by (3.3.a,b,c,d) and Procedure RCC, any projective embedding of L into $E_8(\mathbb{C})$ must result from an AFPF embedding into $E_6(\mathbb{C})$. Such embeddings are ruled out by consideration of E_6-feasible partial characters on the set $\{1A, 2A, 2B\}$ of classes of L; these characters all involve at least one copy of the trivial partial character. We conclude that there are no projective embeddings of L into $E_8(\mathbb{C})$.

Type D. We now consider cases where L has type D_n where $n \geq 4$. The group $D_n(q)$ contains $A_{n-1}(q)$ and so the only candidate is $D_4(2)$, for which an embedding is known. For example, the simple group $D_4(2)$ embeds in the $PSO(8,\mathbb{C})$ subgroup that we consider in Lemma 4.5.

Type B. We now consider groups L with type B_n, where $n \geq 2$. We begin with the case B_2. Since $B_2(q) \geq A_1(q^2)$, we need only consider the cases $q \in \{2,3,4,5,7\}$. The groups $B_2(2)$, $B_2(3)$ and $B_2(5)$ are all known to have projective embeddings. However, the group $B_2(7)$ has a subgroup of the form $7^3.[SO_3(7) \times 3]$. Given an embedding of S into $E_8(\mathbb{C})$, the normal subgroup 7^3 must be toral [GrElAb], which leads to an impossible embedding $SO_3(7) \times 3 \leq W_{E_8}$.

Finally, in the case $L \cong B_2(2^2) \cong Sp(4,4)$, we work with 13 irreducible characters with degrees 1, 18, 34, 34, 50, 51, 51, 85, 85, 153, 204, 204, and 225 restricted to the classes 1A, 2A, 2B, and 2C. The only E_8-feasible partial characters contain the trivial partial character with a multiplicity of between 6 and 11. Thus by (3.3.e,f), there is no Lie primitive embedding of L into $E_8(\mathbb{C})$ or into $E_7(\mathbb{C})$. Now, the smallest faithful projective representation of L has degree 18. Hence by (3.3.a,b,c,d) and Procedure RCC, any projective embedding of L into $E_8(\mathbb{C})$ must give an AFPF embedding into $E_6(\mathbb{C})$. Such embeddings can be ruled out by considering E_6-feasible partial characters on our earlier set of classes; the only feasible characters involve at least 10 copies of the trivial partial character so that (3.3.g) applies. We conclude that there are no projective embeddings of L into $E_8(\mathbb{C})$.

If L has type $B_l(p^n)$, with $l \geq 3$, then the containment $B_l(p^n) \geq A_3(p^n)$ shows that we need only deal with the possibility $p^n = 2$. The group $B_3(2)$ is known to embed. However $B_4(2) \cong Sp(8,2) \geq P S\Omega^-(8,2)$, and there is no restriction of the adjoint character of E_8 to a positive linear combination of partial characters of $PS\Omega^-(8,2)$ on the classes 1A, 2A, 2B, and 2C (six irreducible characters with degrees 1, 34, 51, 84, 204, and 204 must be considered). So, $l = 3$ and $p^n = 2$.

Type C. We consider L with type C_n, where $n \geq 3$. Since $C_n(q)$ contains a group of type $A_3(q)$, we need only consider the case $q = 2$. Then we are in the case $B_n(2) \cong C_n(2)$, which has been treated.

Type ^2D. If S has type $^2D_n(q)$, with $n \geq 4$, then S contains $B_{n-1}(q)$ and therefore can not be embedded, unless perhaps $n = 4$ and $q = 2$. However, as we already noted in considering groups of Type B, $^2D_4(2) \cong PS\Omega^-(8,2)$ is eliminated by a character restriction argument.

Type G_2. If L has type $G_2(q)$, then S must contain central extensions of both $A_2(q)$ and $^2A_2(q)$. Our earlier analysis of these families shows that we only need consider $L \in \{G_2(2), G_2(3)\}$. These two groups are known to be embedded: There is an embedding of $G_2(2)$ in $G_2(\mathbb{C})$, hence in all the exceptional groups. From the character table, we get an embedding of $G_2(3)$ in $SO(14, C)$ hence in any D_7-type subgroup, whence in $2E_7(\mathbb{C})$ and $E_8(\mathbb{C})$.

Types F, E, and ^2E. The group $F_4(q)$ contains $B_4(q)$, the group $^2E_6(q)$ contains $F_4(q)$, and the groups $E_l(q)$ contains $A_5(q)$, so no groups of these types embed projectively in $E_8(\mathbb{C})$.

Type $^3D_4$. Let $L \cong {}^3D_4(q)$, then $S \geq G_2(q)$, and we need only consider the cases $q \leq 3$. The group $^3D_4(2)$ is known to embed. We now show that $^3D_4(3)$ can be eliminated from consideration.

Assume $L \cong {}^3D_4(3)$. Take the parabolic of the form $P = RJ$, where $R = O_3(P) \cong 3^{1+8}$ and $J \cong 2 \times SL(2,27)$. By the $3B$-criterion (3.1.h), a generator z of $Z(P)$ is in $3B$ and so $J' \cong SL(2,27)$ embeds in $C(z) \cong 3A_2E_6$, and in fact embeds in the $3E_6$-factor. In the latter group, the centralizer of an involution has type A_1A_5 or T_1D_5, and no such group can contain a subgroup isomorphic to J', whose low degree nontrivial representations have dimension at least 13.

Type ^2B, the Suzuki groups. Let $L \cong {}^2B_2(q)$. The case of a nontrivial central extension occurs only for $q = 8$ and this will be dealt with below. So, assume S is simple. The order is $q^2(q-1)(q^2+1)$, a Sylow 2-group U of S has order q^2 and $Z(U)$ has order q and is pure. Since $q > 2$ is an odd power of 2, the classification of pure 2-subgroups [CG] implies $q = 8$ or 32. Embeddings for $q = 8$ exist and are classified in [GR8], so we assume $q = 32$. An irreducible for the Borel subgroup has degree at least $(q-1)(q/2)^{\frac{1}{2}}$, which in this case is 124.

Procedure XR, applied to the classes 1A, 2A, 4A shows that the only E_8-feasible character of $Sz(32)$ is the sum of the two distinct 124-dimensional irreducibles, 124 and $124'$. However, $\wedge^2 124$ decomposes as the sum of the six irreducible 1271-dimensional characters of $Sz(32)$ and $124 \otimes 124'$ decomposes as the sum of the fifteen irreducible 1025-dimensional characters together with one copy of the trivial character. It follows that there is no non-trivial invariant anti-commutative algebra structure on a 248-dimensional $Sz(32)$-module with character $124 + 124'$. This shows that L does not embed in $E_8(\mathbb{C})$.

Now suppose that we have $Z(S) \cong 2$ or 2×2 and $L \cong S/Z(S) \cong Sz(8)$. Let U be a Sylow 2-group of L and let B be the normalizer. Let $Z := Z(S)$ and $E := \Omega_1(U)$, rank 4 or 5. Then $E \setminus Z$ is a B-conjugacy class. If F is any complement to Z in E, F is a $2B$-pure eights group. Its normalizer $N := N_G(F)$ has the form $2^4 A_1^8 .2^3 GL(3,2)$ [CG]. It is clear that the image of B in N/N^0 has the form 7 or $2^3{:}7$, whence $U' \leq N^0$. In N^0, every noncentral involution t has the property that it is conjugate in N^0 to some involution of the form tf, for $f \in F, f \neq 1$ [CG]. The fusion pattern in $Z(N^0) \cong 2^4$ which contains F with index 2, is $2A^8 2B^7$. So, elements of Z are noncentral in N^0 and E is $2B$-pure. Then Z contains a $2B$ involution, which puts S in a type D_8-subgroup, a contradiction, since $Sz(8)$ does not projectively embed in $SO(16, \mathbb{C})$.

Type ${}^2F_4(q)'$, the Ree groups Let $L \cong {}^2F_4(q)'$ (${}^2F_4(q)$ is simple, for $q > 2$ and for $q = 2$ its commutator subgroup, of index 2, is the Tits simple group). Since L contains ${}^2B_2(q)$, we only have to consider $q \in \{2, 8\}$. However, L also contains $Sp(4,q)$, which is embedded in $E_8(\mathbb{C})$ only for $q \leq 5$. Hence, we can only have $q = 2$, and there is an embedding for $q = 2$ [CW97].

Type ${}^2G_2(q)'$, the Ree groups For $q = 3$, this is $PSL(2,8)$, which has already been treated. For $q > 3$, the existence of a subgroup $PSL(2,q)$ implies that $q = 27$. We now look at a Borel subgroup B, and let $U := O_3(B)$. Then $A := \Omega_1(U) = Z(U) \cong 3^6$ has too large a rank to be nontoral, but B contains an element of order 13, which does not divide the order of the Weyl group, a contradiction.

Alternating Groups.

We now discuss nonembedding results. The story for embeddings in $G_2(\mathbb{C})$ is covered in [GrG2], so we concentrate on the other exceptional groups.

There is no embedding of $2Alt_{13}$ in $2E_7(\mathbb{C})$ which takes the central involution to the central involution:

An action on a 56-dimensional module with faithful action on each irreducible constituent is impossible for $2Alt_{13}$ since 32 is the the only such irreducible degree less than 56. (Note that $2Alt_{13}$ does embed in $2E_7(\mathbb{C})$ via a subgroup isomorphic to $Spin_{13}(\mathbb{C})$.)

Nonembedding of $2Alt_{18}$ in $E_8(\mathbb{C})$: Suppose $S \cong 2Alt_{18}$. If it embeds in $E_8(\mathbb{C})$, it lies in the centralizer of an involution. If type $2B$, it has a projective representation of degree 16, a contradiction. If type $2A$, it acts on the 56-dimensional irreducible faithfully, a contradiction to the previous paragraph.

Nonembedding of $2Alt_{14}$ in $2E_7(\mathbb{C})$: There is no embedding in which the center goes to the center of $2E_7(\mathbb{C})$, by the earlier consideration of $2Alt_{13}$. Therefore, any embedding lies in the centralizer of a noncentral involution, which has type A_1D_6. and this is clearly not possible by the character table.

Nonembedding of $2Alt_{12}$ in $3E_6(\mathbb{C})$: This follows from the fact that the smallest degree of a faithful character for $2Alt_{12}$ is 32. An embedding in $3E_6(\mathbb{C})$ would give action on the 27-dimensional module for $3E_6(\mathbb{C})$, contradiction.

Nonembedding of $2Alt_{11}$ in $F_4(\mathbb{C})$: If there were an embedding, it would lie in a centralizer of an involution, which has type B_4 or C_3A_1, requiring representations of impossibly small degrees.

Nonembedding of Alt_{11} in $E_8(\mathbb{C})$: Procedure XR shows that there are no E_8-feasible partial characters on the classes 1A, 2A, 2B, 3A, 3B (we need to use eleven irreducible characters of degrees 1, 10, 44, 45, 110, 120, 126, 132, 165, 210, and 231 in this analysis). We now give an alternative, machine free, character theoretic analysis of this case. Suppose that there is a subgroup $L \cong Alt_{11}$ of $E_8(\mathbb{C})$. We consider the restriction of the adjoint module to it.

We observe that the degree 126 character may not occur since it is nonreal (if it were to occur, it would occur with multiplicity at least 2, a contradiction since $2 \cdot 126 > 248$).

The irreducible degrees less than 249 (excluding degree 126) and their remainders modulo eleven are

1	10	44	45	110	120	132	165	210	231
1	10	0	1	0	10	0	0	1	0

Let b, c be the number of irreducible constituents for L of degree congruent to 1 and 10, respectively. Notice that $248 \equiv 6(mod\ 11)$.

Suppose $c \geq 5$. Then we have a sum of c irreducibles of degrees 10 and 120 of total degree at most 248. Let t be the involution of class $2A$ in L. These irreducibles would contribute at least $5 \cdot 6 = 30$ to its trace. In $E_8(\mathbb{C})$, involutions have trace -8 and 24 only, so we need an irreducible with negative trace, and this means dimension 210, too large, a contradiction.

We now have $c \leq 4$. The congruence $248 \equiv 6(mod\ 11)$ implies that $b \geq 6$. The trivial character does not occur here or else such an embedding in $E_8(\mathbb{C})$ centralizes a 1-torus. We show elsewhere in this section that Alt_{10} is not in $2E_7(\mathbb{C})$, so this 1-torus must have centralizer with components of types A and D only. Thus, any involution t in this 1-torus must be in class $2B$. So, in the corresponding 16-dimensional projective representation of $C(t) \cong HSpin(16, \mathbb{C})$, L must have constituents of dimensions 10, 1,1,1,1,1,1. Then in $C(t)$, we have the double cover $2Alt_{11}$, not a simple group, a contradiction. The $b \geq 6$ constituents therefore contribute a degree of at least 6×45, which exceeds 248, a contradiction.

Nonembedding of Alt_{10} in $2E_7(\mathbb{C})$: Procedure XR shows that there are no E_7-feasible partial characters on the classes 1A, 2A, 2B, 3A, 3B (we need to use nine irreducible characters of degrees 1, 9, 35, 36, 42, 75, 84, 90, and 126 in this analysis). For a machine free argument, we consider the trace of a noncentral involution of $2E_7(\mathbb{C})$ on the degree 56 irreducible is ± 8. This can not be achieved by a degree 56 character of Alt_{10}.

Nonembedding of Alt_8 in $3E_6(\mathbb{C})$: Procedure XR for $3E_6$ shows that there is no feasible restriction of the 27-dimensional module to partial characters on the classes 1A, 2A, 2B (we need to use six irreducible characters of degrees 1, 7, 14, 20, 21, and 21 in this analysis). For a machine free argument, we consider the degree 27 irreducible for $3E_6(\mathbb{C})$. The allowed traces for involutions are 7 and -5. Note that on all representations of degree at most 27 for Alt_8, the class $2B$ has positive trace. Any irreducible for Alt_8 which occurs here has degree at most 21. Suppose a degree 21 irreducible for Alt_8 occurs. This contributes 1 to the trace of $2B$, so to achieve trace 7, we need 6 copies of the trivial irreducible, giving trace 3 on $2A$, contradiction. If we have a degree 20 irreducible, we need either a degree 7 irreducible or 7 copies of the trivial irreducible, again giving a contradiction with the trace of $2A$. Suppose only degrees 14 or less occur in the 27 dimensional representation. Then we can not arrange for trace 7 on class $2B$, contradiction.

Nonembedding of Alt_7 in $F_4(\mathbb{C})$: Let J be the degree 26 irreducible and S a putative Alt_7-subgroup of $F_4(\mathbb{C})$. The allowed traces for an involution of $F_4(\mathbb{C})$ on J are -4 and 6 (compare with the traces for $2E_7(\mathbb{C})$ involutions on the degree 27 irreducible). The only negative traces on involutions for irreducibles of S of degree less than 27 are trace -2 at two 10-dimensional irreducibles and trace -1 on the degree 15 irreducible. Therefore, involutions of S have trace 6. The module J for $F_4(\mathbb{C})$ is self-dual, so if a degree 10 irreducible occurs, a second must occur since they are nonreal. If this happens, trace 6 on involutions is impossible, so no degree 10 irreducible occurs.

The allowed traces for elements of order 3 of $F_4(\mathbb{C})$ on J are $-1, 8, 8$. The only irreducible of degree less than 27 which is negative on an element of order 3 is of degree 14. So, elements of order 3 have trace 8.

If a degree 15 irreducible occurs, the other constituents have degrees 1 and 6 only. In this case, we can not get trace 8 on class $3B$.

We conclude that only degrees 14, 6 and 1 may occur. The trace 6 requirement for involutions forces the decomposition $26 = 14 + 6 + 6$ and we fail to get trace 8 on class $3B$, a final contradiction to existence of an embedding of $2Alt_7$ in $F_4(\mathbb{C})$.

Sporadic simple groups. Procedure XR shows that M_{22} can not have fixed point free actions on the adjoint modules of $E_8(\mathbb{C})$, $E_7(\mathbb{C})$ or $E_6(\mathbb{C})$. (We use the classes 1A, 2A, 3A of M_{22}, and irreducible characters of degrees 1, 21, 45, 55, 99, 154, 210, and 231 in this application of Procedure XR.) Therefore, since M_{22} has no projective representation with degree less than 21 except for nonorthogonal degree 10 representations, Procedure RCC and (3.3a,b,c) show that there are no projective embeddings of M_{22} into $E_8(\mathbb{C})$. As a corollary, we can also rule out projective embeddings of the sporadic simple groups

$$\{M_{23}, M_{24}, HS, McL, Co_3, Co_2, Co_1, Fi_{22}, Fi_{23}, Fi_{24}, HN, Ly, J_4, F_2, F_1\}$$

into $E_8(\mathbb{C})$ (because each of these groups contains M_{22} or one of its covers).

Procedure XR shows that there are no E_8-feasible partial characters of J_1 on the classes 1A, 2A, 3A. Since J_1 has trivial multiplier, this shows that there are no projective embeddings of J_1 into $E_8(\mathbb{C})$, moreover, since $J_1 \leq ON$, there are no projective embeddings of ON into $E_8(\mathbb{C})$.

The Rudvalis group Ru contains a copy of $PSU(3,5)$ and therefore it can not be projectively embedded into $E_8(\mathbb{C})$. Similarly the Held group He contains a copy of $Sp(4,4)$, and the Suzuki group Suz contains a copy of $G_2(4)$, so that these two sporadic groups can not be projectively embedded into $E_8(\mathbb{C})$.

Procedure XR shows that there are no AFPF embeddings of J_3 into the algebraic groups $E_8(\mathbb{C})$, $E_7(\mathbb{C})$, and $E_6(\mathbb{C})$. (We use the classes 1A, 2A, 3A, 3B of J_3, irreducible characters of degrees 1, 85 of J_3, and we need to consider feasible restrictions of the adjoint characters of E_8, E_7 and E_6.) Therefore, since J_3 has no faithful projective representation with degree less than 18, Procedure RCC shows that there are no projective embeddings of J_3 into $E_8(\mathbb{C})$.

There is only one E_8-feasible character of F_3 (since the only possible faithful irreducible constituent has degree 248). Since this representation does not support an invariant alternating trilinear form, F_3 is not projectively embedded in $E_8(\mathbb{C})$.

We have now shown that 23 of the 26 sporadic simple groups are not projectively embedded into $E_8(\mathbb{C})$. Projective embeddings of the remaining sporadics, M_{11}, M_{12}, and HJ are known [CG], via subgroups of types D_5, B_5 and A_5.

4. Table QE entries.

We shall go over Table QE, row by row, but first we start with a few preliminaries about conjugacy and primitivity and some technical results.

Some Conjugacy Results.

(4.1) Definition. If $H < G$ are groups, we say that H *strongly controls fusion of its subsets with respect to* G if, given subsets A and B of H and $g \in G$ such that $A^g = B$, there is $h \in H$ so that $a^g = a^h$, for all $a \in A$. In other words, g can be factorized $g = ch$, for some $c \in C_G(A)$ and $h \in H$. Similarly, we define strong control of fusion of subsets which satisfy certain conditions.

(4.2) Proposition. (i) $O(n, \mathbb{C})$ and $Sp(n, \mathbb{C})$ (when n is even) strongly control fusion of subsets with respect to $GL(n, \mathbb{C})$.

(ii) $SO(n, \mathbb{C})$ strongly control fusion of subsets with respect to $GL(n, \mathbb{C})$ for subsets which centralize an orthogonal transformation of determinant not 1. If n is odd, $SO(n, \mathbb{C})$ strongly control fusion of subsets with respect to $GL(n, \mathbb{C})$.

(iii) $G_2(\mathbb{C})$ strongly controls fusion of its subsets with respect to $GL(7, \mathbb{C})$.

(iv) $F_4(\mathbb{C})$ strongly control fusion of subsets with respect to $3E_6(\mathbb{C})$.

Proof. See [GrG2]; (i) and (ii) are old results. □

(4.3) Remark. A consequence of (4.2.iii) is that the $G_2(\mathbb{C})$ column of Table QE will be definitive in counting conjugacy classes of embeddings since the relevant degree 7 characters of embeddings are known. See [CW83] and the revisions in [GrG2], which explicitly settles conjugacy of embeddings of all quasi-simple groups which are irreducible on the degree 7 module. The reducible cases are treated as they arise in this section. The above result (4.2.iv) from [GrG2] was a pleasant corollary of the method for the main results about $G_2(\mathbb{C})$. It has been used in the work of Frey on subgroups of $F_4(\mathbb{C})$ and $3E_6(\mathbb{C})$.

(4.4) Application. There are many cases in Section 4 of quasisimple groups S in $E_8(\mathbb{C})$ which contain an involution of type $2B$ in the center. Any such S lies in a subgroup $H \cong HSpin(16, \mathbb{C})$. If two such are isomorphic and correspond to perfect subgroups of $SO(16, \mathbb{C})$ which are isomorphic and have the same degree 16 character, then these subgroups are conjugate in H if they satisfy condition (4.2.ii).

(4.5) Lemma. Let $E \leq E_8(\mathbb{C})$, $E \cong 2^2$ with type BBB. Then $C_G(E) \cong 2^2 D_4^2 : 2$. If t is an involution in $C(E) \setminus C(E)^0$, then $\langle E, t\rangle \cong 2^3$ with class distribution $1A^1 2A^1 2B^6$, and $C(\langle E, t\rangle)^0 \cong PSO(8, \mathbb{C})$.

Proof. Since t interchanges the two D_4-factors of $C(E)^0$, $C(\langle E, t\rangle)^0$ has type D_4 and is in fact $PSO(8, \mathbb{C})$ [CG]. If $\langle E, t\rangle$ has a elements from class $2A$ and b elements from class $2B$, then summing traces gives $8 \times dim(D_4) = 8 \times 28 = 248 + 24a - 8b$. Since $a + b = 7$, we deduce $a = 1$ and $b = 6$. □

(4.6) Lemma. Let $E \leq E_8(\mathbb{C})$, $E \cong 2^2$ with type ABB. Then $C_G(E) \cong T_1A_7$. If L is an A_7 factor, $L \cong SL(8,\mathbb{C})/\{\pm 1\}$.

Proof. The shape of $C_G(E)$ comes from [CG]. Let $t \in E \cap 2A$; $C(t) \cong 2A_1E_7$. Then $L \leq M$, a $2E_7$ factor. Since $rk(L) = rk(M)$, a maximal torus T of L is a maximal torus of M. If Q is the root lattice for M, L is generated by the root groups associated to a sublattice $R \leq Q$ of type A_7. From [GrElAb], $L \cong \widetilde{L}/Z$, where $\widetilde{L}$ is the simply connected group of type A_7 and $Z \cong Q/R \cong Z_2$. □

(4.7) Proposition. Let S be a quasisimple finite group, $Z \leq Z(S)$, $S/Z \cong \Omega^+(8,2)$ and suppose $H \leq E_8(\mathbb{C})$, $H \cong 2E_7(\mathbb{C})$.

(a) If $S \leq H$, then $Z = 1$ or $Z \cong 2^2$, and S lies in a natural $2^2D_4(\mathbb{C})$ subgroup of H. In particular, $S \cap Z(H) = 1$.

(b) If $Z = 1$ or $Z \cong 2^2$, then S embeds in a conjugate of H.

Proof. (a) Let $S \leq H$ and $Z \neq 1$. Assume first that Z is not contained in $Z(H)$. Then S lies in $C_H(t)$, for some noncentral involution t of H. From [CG] or [GrElAb], $C_H(t)$ has type T_1E_6 or A_1D_6. In the former case, S lies in the $3E_6$-factor, so in an involution centralizer, of type simply connected D_5T_1 or A_1A_5. In either case, S lies in a natural D_4-subgroup, forcing $Z \cong 2^2$ and $Z \cap Z(H) = 1$.

It remains to treat the case $Z \cong 2$, $Z = Z(H)$. On the 56-dimensional irreducible for H, noncentral involutions have trace ± 8, but this is impossible for a sum of faithful irreducibles of S.

(b) For $Z = 1$, use Corollary 4.5 and for $Z \cong 2^2$, look in a natural D_4-subgroup of H. □

(4.8) Remark. For finite simple and quasisimple subgroups of $F_4(\mathbb{C})$ and $3E_6(\mathbb{C})$, see [CW97], p. 140 and p. 140-141, respectively. The simple members of the list for $3E_6(\mathbb{C})$ which lie in $F_4(\mathbb{C})$ are listed in Corollary 7.3 [CW97], p. 141. It is a relatively simple matter to decide which finite quasisimple non-simple groups on this list lie in $F_4(\mathbb{C})$ since they lie in the centralizer of a nontrivial element of finite order and so are in a subgroup of type corresponding to a proper subset of the extended Dynkin diagram for $F_4(\mathbb{C})$. All such are central products of groups of classical type; hence the character table and power maps settle existence of the embeddings (1.1).

Some Primitivity Results.

There are a few ways a subgroup S of the algebraic group G can fail to be primitive.

(4.9) Definition. The Lie imprimitive subgroup S of G is called *intrinsically imprimitive* if S lies in a *connected* positive dimensional group.

Suppose that the imprimitive group S is not intrinsically imprimitive. Then S lies is in a positive dimensional subgroup but for every such subgroup, S does not lie in the connected component of the identity; it permutes the central factors by conjugation. The second way does happen (e.g. $S \cong GL(3,2)$ in a subgroup of type A_1^7 in $E_7(\mathbb{C})$) but the intrinsic way is the more usual situation for quasisimple S. We want a result to show that the first case must hold when we want to show that a group S is Lie primitive by seeking a contradiction. The lemma below shows how to replace S by a subgroup U which lies in a proper connected positive dimensional group; a contradiction to the existence of U would imply a contradiction to the non-Lie primitivity of S.

(4.10) Definition. A finite subgroup F of the connected Lie group G is *tame* if
(i) any proper subgroup of F has index greater than $max\{3, rank(G)\}$; and
(ii) F is fixed point free on the adjoint module;
(iii) an F-invariant subspace of the adjoint module is not a toral subalgebra.

(4.11) Remarks. (i) A subgroup F is tame if any F-irreducible constituent on the adjoint module has dimension greater than $rank(G)$. In $E_8(\mathbb{C})$, if F is isomorphic to $PSL(2,q)$ or $SL(2,q)$ for $q \geq 11$, then F is tame if it is fixed point free on the adjoint module (for if M is an irreducible constituent of dimension at most 8, then if M were a toral subalgebra, F would act nontrivially on a corresponding toral subgroup and so $PSL(2,q)$ would be involved in the Weyl group of E_8, which is not so).

(ii) If a finite group F is not tame, then it has a nontrivial homomorphism to Sym_8. So, if F is quasisimple, $F/Z(F) \cong PSL(2,7)$ or Alt_n, for $n \leq 8$.

(4.12) Lemma. Let F be a tame subgroup of of the quasisimple algebraic group G. If F is not Lie primitive, then there is a proper closed connected positive dimensional semisimple subgroup H containing a subgroup U such that $U/Z(H^0) \cap U$ is a nontrivial quotient group of F.

Proof. Let H be a proper closed positive dimensional subgroup containing F. We may assume that H is reductive. Since F has no proper subgroup of small index, F normalizes all quasisimple components of H^0. Since $C_G(S)$ is 0-dimensional and F has no quotient isomorphic to any group of graph automorphisms, F acts nontrivially as inner automorphisms on all quasisimple components of H. Just take U to be a subgroup of H^0 inducing that group of automorphisms under conjugation.

(4.13) Corollary. When S is a covering of a simple group, so is U', where U is the group described by the previous Lemma. Furthermore, we have $S = U'$ when $C_G(H)$ is abelian.

The Corollary describes the situation where we use the intrinsic Lie imprimitive concept.

(4.14) Definition. We call the group U' of (4.13) an *intrinsic associate* for S. It exists whenever S is Lie imprimitive and satisfies the conditions of (4.10.i).

(4.15) Remark. If S fails to have an intrinsic associate, then $S/Z(S) \cong PSL(2,7)$ or Alt_n, for $n \leq 8$, a limited set of groups.

(4.16) Convention. If the quasisimple group S is Lie imprimitive, an intrinsic associate is intrinsically Lie imprimitive even if S is not. It is a convenience to have such an S is intrinsically Lie imprimitive when searching for a contradiction to prove that any such S is Lie primitive. We shall use this property for all L but the ones in (4.15). Among all cases of (4.15), we know of no example of an S primitive in an exceptional group G except for $PSL(2,7)$ in $G_2(\mathbb{C})$.

Details about the entries of Table QE.

We now proceed to justify the entries of Table QE in detail, in groups of rows by isomorphism type of the finite simple group in column 2. We give what we know about existence, fusion patterns, conjugacy of embeddings, characters and primitivity of embeddings. In only a few cases is there a nearly complete classification of embeddings up to conjugacy. In particular, we determine the exact list of QE-pairs and so justify the Main Theorem claimed in the introduction. Table QE summarizes the results of Sections 3 and 4, but may not refer to every detail in these sections.

Throughout the remainder of Section 4, L denotes a finite simple group, S denotes a perfect central extension of it. We consider embeddings of S in some exceptional algebraic group, G, and we need to consider only cases for L not eliminated in Section 3, the cases listed in column 2. Proofs available in existing literature are noted in the fifth column.

Note that a map to an exceptional group in one column gives a map to all columns to the right. Some rows need not be discussed if they involve a noted exceptional isomorphism.

Rows for Alt_5 in Table QE.

Let $L := Alt_5$. First, we treat the story of maps to $G_2(\mathbb{C})$.

For $G_2(\mathbb{C})$, we use Theorem 2 of [GrG2] which tells us that S is in a local subgroup since it is reducible on the 7-dimensional module. Since S is local, it is in a natural $3A_2$ or $2A_1A_1$.

Let us use the irreducible degrees as usual for irreducible representations of S. We may put primes on the degree if there is more than one irreducible in a given dimension. So the irreducibles for L are $1, 3, 3', 4$

and 5 and the additional irreducibles for S are $2, 2', 4'$ and 6. They satisfy $2 \otimes 2' = 4$, $2 \otimes 2 = 3 + 1$ and $2' \otimes 2' = 3' + 1$.

We need to discuss an involution centralizer in $G_2(\mathbb{C})$. It has the form $S_1 \circ S_2$, where each factor is a fundamental $SL(2, \mathbb{C})$. We arrange notation so that the first factor is associated to long roots and the second factor is associated to short roots. On the 7-dimensional module, the first factor has irreducibles of dimensions 2,2,1,1,1 and the second factor has irreducibles of dimensions 2,2,3.

We claim that S is in an involution centralizer in $G_2(\mathbb{C})$. If not, $S \cong Alt_5$ is in a $3A_2$ subgroup and we have decomposition $1 + 3 + 3$ or $1 + 3' + 3'$, which means that S is in the centralizer of an involution which acts as a graph automorphism on the $3A_2$ subgroup. So, the claim is proven. So, let $S \leq C(z)$, for $|z| = 2$. If S is simple, the -1 eigenspace of z has the form $2 \otimes 2$ or $2 \otimes 2'$, $2' \otimes 2$ or $2' \otimes 2'$, according to the two factors of the involution centralizer as discussed above. These lead to respective irreducible decompositions $3 + 1 + 3$, $4 + 3$, $4 + 3'$ and $3' + 1 + 3'$. If $S \cong SL(2, 5)$, it lies in one factor of an involution centralizer of $G_2(\mathbb{C})$, whence decompositions $2 + 2 + 1 + 1 + 1$, $2' + 2' + 1 + 1 + 1$ or $2 + 2 + 3$ or $2' + 2' + 3'$. So, we get 4(2) classes for each of Alt_5 and $SL(2, 5)$. This completes the story for $G_2(\mathbb{C})$.

For conjugacy of embeddings of Alt_5 and $SL(2, 5)$ into $F_4(\mathbb{C})$, $3E_6(\mathbb{C})$ and $E_8(\mathbb{C})$, see [Fr1], [Fr2], [Fr3], [Fr4]. Interestingly, the difficult ZDC case for embeddings of Alt_5 in $E_8(\mathbb{C})$ does not occur for embeddings in $2E_7(\mathbb{C})$. It follows that there are no primitive embeddings in $E_7(\mathbb{C})$ and any that are primitive in $E_8(\mathbb{C})$ must be in the ZDC situation, the unique unresolved case for embeddings in $E_8(\mathbb{C})$ for which only one class is known, represented by an Alt_5 in a $PSL(2, \mathbb{C})$-subgroup.

Rows for Alt_6 in Table QE.

In $G_2(\mathbb{C})$, any projective embedding must be that of $3{\cdot}Alt_6$ in the $3A_2$ subgroup. The character table shows that only $3 + 3 + 1$ and $3' + 3' + 1$ are possible as decompositions into irreducibles. See [CW83] and [GrG2].

The group $2Alt_6$ embeds in $SL(4, \mathbb{C})$, so we get embeddings in $F_4(\mathbb{C})$ and overgroups.

We show nonembedding of $6Alt_6$ in $F_4(\mathbb{C})$. If there were an embedding, an element of order 3 in the center of $6Alt_6$ must lie in the class in F_4 whose centralizer has shape $3A_2A_2$; but then $6Alt_6$ would embed in $SL(3, \mathbb{C})$, which is impossible.

In $E_8(\mathbb{C})$, any embedding of $3Alt_6$ must be in a natural $3A_2E_6$ type subgroup by (3.1.h), and any embedding of $6Alt_6$ must be in a natural $6A_5$ subgroup in a natural $3E_6$. Therefore, both of these central extensions of Alt_6 embed in a $3E_6(\mathbb{C})$ subgroup and overgroups. For $6Alt_6$, we have just two degree 6 characters, fused by outer automorphisms, hence table entry 2(1) in column $3E_6(\mathbb{C})$.

Consider $S \cong 3Alt_6$ as a subgroup of $F_4(\mathbb{C})$. Let z generate $Z(S)$. Then $z \in C(z)'$ implies that $S < C(z) \cong 3A_2A_2$. Let H_1, H_2 be the two central factors. Each is normal in $N(C(z))$. Form the external direct product $H := H_1 \times H_2$, and let z_i generate the center of H_i and choose notation so that $z_1 z_2^{-1} = 1$ in $C(z)$. The embedding of S in $C(z)$ factors through an embedding in H, which is described by a pair of homomorphisms f_i to H_i. The character table of S allows just four inequivalent embeddings in $SL(3,\mathbb{C})$, which form a single orbit under $Aut(S)$ and two orbits under the inverse transpose of $SL(3,\mathbb{C})$, which is just what $N(C(z))/C(z)$ induces on both factors. We have three cases: $Ker(f_1) = 1 \neq Ker(f_2)$, $Ker(f_1) \neq 1 = Ker(f_2)$, and $Ker(f_1) \neq 1 \neq Ker(f_2)$. For each f_i, we have five choices, but the pair must be chosen so that the image of $3Alt_6$ under the composite of (f_1, f_2) to $H \cong SL(3,\mathbb{C}) \times SL(3,\mathbb{C})$ followed by the quotient map to $C(z)$ is isomorphic to $3Alt_6$. These respective cases lead to $[1 \cdot 4 + 4 \cdot 1]/2 + [4 \cdot 2/2] = 8$ $F_4(\mathbb{C})$-classes of embeddings, which form $1 + 1 + 2 = 4$ classes by further equivalence under $Aut(S)$.

It is clear that we get embeddings of Alt_6 in $F_4(\mathbb{C})$, $3E_6(\mathbb{C})$, $2E_7(\mathbb{C})$ and $E_8(\mathbb{C})$ via a subgroup $3A_2^2$ in $F_4(\mathbb{C})$ since the two factors intersect in a group of order 3. We do not claim that an embedding is conjugate to one of these. For maps like these, we account for conjugacy by counting as in the last paragraph. Here, only the third case is relevant and we require the image of $3Alt_6$ in $C(z)$ to be isomorphic to Alt_6. The number of $F_4(\mathbb{C})$-classes of appropriate pairs (f_1, f_2) for this case is $(4 \cdot 2)/2 = 2$ and this reduces to 1 by $Aut(S)$-equivalence.

Rows for Alt_7 in Table QE.

Nonembedding of $6Alt_7$ in $F_4(\mathbb{C})$ follows from the $6Alt_6$ result. For $3Alt_7$, any embedding in $F_4(\mathbb{C})$ would have to factor through a subgroup of type $3A_2^2$. However, $3Alt_7$ has no representations of degree 3, a contradiction. Because of $6A_5(\mathbb{C}) < 3E_6(\mathbb{C})$, we have embeddings of Alt_7, $3Alt_7$ and $6Alt_6$ in $3E_6(\mathbb{C})$. Also, $2Alt_7$ embed in $SL(4,\mathbb{C})$, so we get embeddings in $F_4(\mathbb{C})$ and overgroups. Nonembedding of Alt_7 in $F_4(\mathbb{C})$ was established in our consideration of Alternating groups in Section 3.

The nonembedding of $2Alt_7$ in $G_2(\mathbb{C})$ follows from the structure of an involution centralizer, $2A_1^2$.

Rows for Alt_n, $n \geq 8$ in Table QE.

In Section 3, we presented some nonembedding results for these alternating groups and covers. Here, we give embedding results for all the cases not eliminated earlier. Compare with the treatments in [CW95][CG].

In a natural $HSpin(16,\mathbb{C})$-subgroup, we have an embedding of $2Alt_{17}$ and in a natural A_8-subgroup of $E_8(\mathbb{C})$, we have an embedding of Alt_{10}.

Analogously, we have an embedding of $2Alt_{13}$ in $E_7(\mathbb{C})$ via a natural 2^2D_6-subgroup and an embedding of Alt_9 via a type A_7-subgroup.

Also, we get an embedding of $2Alt_{11}$ in $3E_6(\mathbb{C})$ via a $4D_5$-subgroup.

For $F_4(\mathbb{C})$, we have an embedding of $2Alt_{10}$ via a natural B_4-subgroup.

Rows for $A_1(7)$ in Table QE.

Let $L := PSL(2,7)$. Note that $PSL(2,7)$ lies in any A_2-subgroup of $E_8(\mathbb{C})$, so we have embeddings in all exceptional groups. Note that $SL(2,7)$ embeds in the simply connected group $4A_3$, so we get embeddings in $F_4(\mathbb{C})$ and overgroups.

Consider maps to $G_2(\mathbb{C})$. The only projective embeddings are embeddings of L since an involution centralizer in $G_2(\mathbb{C})$ has shape $2A_1^2$. Theorem 2 of [GrG2] puts a reducible S in a natural $3A_2$-subgroup, whence the character must be $1+3+3'$, and the two such embeddings in the $3A_2$ are conjugate in the normalizer. So, we get one class this way. For the irreducible case, see [GrG2] which shows we get one class.

For $E_8(\mathbb{C})$, see [K], which gets the number of conjugacy classes for embeddings of both $PSL(2,7)$ and $SL(2,7)$, for almost all the fusion patterns.

Rows for $A_1(8)$ in Table QE.

Let $L := PSL(2,8)$. We get a family of three algebraically conjugate embeddings in $G_2(\mathbb{C})$ [CW83], [GrG2].

Rows for $A_1(11)$ in Table QE.

Let $L := PSL(2,11)$. By (3.1.a) and the order of the Weyl group of F_4, we get no projective embeddings in $F_4(\mathbb{C})$ (the original proof showed character infeasibility by a computation, p. 142, [CW97]). No projective embeddings therefore exist to $G_2(\mathbb{C})$. There is an embedding in any type A_4 subgroup, hence also in D_5 and any E_n. Note that $SL(2,11)$ embeds in both $4D_5$ and $SL(6,\mathbb{C})$ hence in $3E_6(\mathbb{C})$ and overgroups.

Rows for $A_1(13)$ in Table QE.

Let $L := PSL(2,13)$. For $G_2(\mathbb{C})$, there are just two characters of degree 7, algebraically conjugate, hence two classes of embeddings. This gives Table QE entry 2(1). At once, we get at least one class of embeddings of L in $F_4(\mathbb{C})$.

For $3E_6(\mathbb{C})$, we have three classes of subgroups described in [CW93]. In [CW97], Table 8, we get three possible degree 27 characters, $6{\cdot}1+3{\cdot}7_x$ (2 algebraically conjugate classes), 13_a+14_a (just 1), $1+12_x+14_a$, (3 algebraically conjugate classes, since there are three algebraically conjugate degree 12 characters); by (7.3) [CW97], these are the only ones. Note that 14_b does not occur here because an involution in S has trace -5 or 3 on the degree 27 module for $3E_6$. So, we get 6 characters which fall in 3 orbits by additional algebraic conjugacy. From [CW97], 7.1, and [CW93], members of the second and third families of these characters correspond to just one conjugacy class of embeddings, so we get at least 6(3) classes for $3E_6(\mathbb{C})$ and 3(2) classes for $F_4(\mathbb{C})$; both known members of the first family are realized by an embedding in a natural $G_2(\mathbb{C})$ subgroup, for which we have already discussed conjugacy. If every embedding from the first family comes from a natural $G_2(\mathbb{C})$ subgroup, then we have exactly 6(3) classes; but we belive that this is unsettled.

In $F_4(\mathbb{C})$, there is an involution centralizer of shape $2A_1C_3$, giving an embedding of $SL(2,13)$ by its 6-dimensional representation. Observe also that since $6A_5$ is in $3E_6(\mathbb{C})$, we get an embedding of $SL(2,13)$ in $3E_6(\mathbb{C})$ and overgroups. The other involution class in $3E_6(\mathbb{C})$ has centralizer shape T_1D_5, which has no subgroup isomorphic to $SL(2,13)$. For the embeddings of $SL(2,13)$ described above, the central involution z is in $2A$ [GrElAb]. Since there are just two faithful degree 6 representations of S, these remarks and (4.2.i) show that there are exactly two classes of embeddings of $SL(2,13)$ in $F_4(\mathbb{C})$ and in $3E_6(\mathbb{C})$.

The structure of an involution centralizer $2A_1^2$ for $G_2(\mathbb{C})$ shows that $SL(2,13)$ does not embed in $G_2(\mathbb{C})$.

Rows for $A_1(16)$ in Table QE.

Here is a proof of the nonembedding of $PSL(2,16)$ in $2E_7(\mathbb{C})$, which also implies nonembedding in $G_2(\mathbb{C})$, $F_4(\mathbb{C})$ and $3E_6(\mathbb{C})$). Take a Borel subgroup B of $PSL(2,16)$ and let $U := O_2(B)$. In $E_8(\mathbb{C})$, U must be $2B$-pure and is unique up to conjugacy, with $C(U)$ of shape $T_8{:}2^{1+6}$ and $N(U)$ of shape $T_8{:}2^{1+6}GL(4,2)$ (T_8 denotes a rank 8 torus). Let F be a cyclic group of order 15 in B. Then, $C_{C(U)}(F) = \langle t \rangle$, where t inverts T_8 under conjugation. This is a contradiction since $C(U) \cap C(F) = C(B)$ contains the natural A_1 subgroup centralizing $2E_7(\mathbb{C})$. (Nonembedding of $PSL(2,16)$ in $F_4(\mathbb{C})$ was proved in [CW97], Proof of Theorem 7.1, p. 141) .

The group L does embed in any type B_7 subgroup, hence in D_8 and E_8.

Rows for $A_1(17)$ in Table QE.

Let $L := PSL(2,17)$. There is no projective embedding into $G_2(\mathbb{C})$[CW83][GrG2]. For $F_4(\mathbb{C})$, there is an algebraically conjugate set of two primitive embeddings (unique up to conjugacy, p.132 [CW97]) and another algebraically conjugate set of two embeddings of $PSL(2,17)$ in a natural $2B_4$-subgroup. Hence, we get at least 4(2) classes of embeddings of $PSL(2,17)$ into $F_4(\mathbb{C})$. Consequently, we have embeddings of $PSL(2,17)$ into overgroups.

Any embedding of $PSL(2,17)$ into $3E_6(\mathbb{C})$ lies in a natural $F_4(\mathbb{C})$ subgroup, by Section 7 and Table 8, p.140 [CW97].

There is an embedding of $SL(2,17)$ into $SL(8,\mathbb{C})$, seen within a natural $3A_8$ subgroup, and this $SL(8,\mathbb{C})$ subgroup lies in a natural $2D_8$ subgroup (reason: it centralizes an involution corresponding to $diag(-1^8, 1^1)$ in $SL(9,\mathbb{C})$, and this is a $2B$ involution in $E_8(\mathbb{C})$). This gives at least 2(1) for the $E_8(\mathbb{C})$ column.

We now argue that there is no embedding of $SL(2,17)$ into $2E_7(\mathbb{C})$. Suppose there were an embedding. The 56-dimensional module decomposes as a sum of faithful characters of $SL(2,17)$ which have degrees in the set $\{8,16,18\}$. However, since 56 is a multiple of 8, any 18-dimensional constituents must occur in batches of four, which is impossible since $4 \times 18 > 56$. Now an element, g say, of $SL(2,17)$ with order 17 has no fixed point in any faithful representation with degree 8 or 16. Hence, g has no fixed vectors in the 56-dimensional module. A similar argument shows that for some m with $0 \leq m \leq 3$, the multiplicity of any other eigenvalue of g is either m or $7-m$. An enumeration of elements of order 17 in $2E_7(\mathbb{C})$ by EFO theory shows that there are no such elements. Therefore, there is no such embedding.

Rows for $A_1(19)$ in Table QE.

Let $L := PSL(2,19)$. By (3.1.a), if L projectively embeds in an exceptional group, it is of type E_n, for some n. If it were to embed in a classical type subgroup, that group must have type A_8 (or rank at least 9 if into a group preserving a bilinear form, and this is impossible). Therefore, any imprimitive embedding of S factors through an embedding into a natural $3A_8$-subgroup or a smaller exceptional subgroup. We deduce that an embedding of S in $3E_6(\mathbb{C})$ is Lie primitive and for such an embedding, $S \cong PSL(2,19)$.

In [CW97], Section 7.6 contains a construction of an embedding with character 9+18 on the degree 27 irreducible for $3E_6(\mathbb{C})$.

The occurrence of $SL(2,19)$ in $2E_7(\mathbb{C})$ is due to Serre [S96]; he embeds the simple $PSL(2,19)$ in the adjoint type group $E_7(\mathbb{C})$ and shows that it lifts to $SL(2,19)$ in $2E_7(\mathbb{C})$ (see [S96], p. 550). From the remarks above, this is primitive.

Rows for $A_1(25)$ in Table QE.

For $PSL(2,25)$, [CW97] Section 6.6 proves existence in $F_4(\mathbb{C})$. We get at least 1(1). Embeddings in overgroups of $F_4(\mathbb{C})$ follow. Clearly, there are no embeddings in $G_2(\mathbb{C})$ [CW83][GrG2].

We now prove that the group $SL(2, 25)$ does not embed in $2E_7(\mathbb{C})$. Let $S \cong SL(2, 25)$, $S < H \cong 2E_7(\mathbb{C})$. Assume first that $Z(S) = Z(H)$. The faithful characters of $SL(2, 25)$ have degrees 12, 24, and 26. These degrees are congruent to either 2 or 0 modulo 12. Now 56 is congruent to 8 modulo 12, and therefore any decomposition of 56 as a sum of faithful character degrees of S must involve at least four copies of 26, a contradiction. If $Z(S) \neq Z(K)$, then S is in an involution centralizer of $E_7(\mathbb{C})$. In this case we would obtain an embedding of S into a group of one of the types A_7 and A_1D_5. Both of these cases would lead to impossibly small degree representations of $SL(2, 25)$.

We now classify embeddings of $SL(2, 25)$ in $E_8(\mathbb{C})$. The last paragraph implies that any embedding of S in $E_8(\mathbb{C})$ puts the central involution in the $2B$ class. The character table of $SL(2, 25)$ shows that this occurs only by a nonfaithful irreducible of degree 13, of which there are two. Since the trivial constituent on the 16 dimensional representation occurs with multiplicity 3, the image centralizes a reflection, so we have exactly 2 classes of embeddings in the $HSpin(16, \mathbb{C})$-subgroup, hence the same number in $E_8(\mathbb{C})$. These two classes fuse under $Aut(S)$, hence entry 2(1) in the $E_8(\mathbb{C})$-column.

Rows for $A_1(27)$ in Table QE.

For $PSL(2, 27)$, [CW97], 6.5 shows that there is an embedding of the simple group in $F_4(\mathbb{C})$. We get at least 3(1). Since an elementary abelian 3-subgroup of $G_2(\mathbb{C})$ has rank at most 2, the group and its double cover do not embed in $G_2(\mathbb{C})$.

Embeddings of the group $SL(2, 27)$ are treated in [GR27]. There are twelve $2E_7(\mathbb{C})$-classes of embeddings and then two by $Aut(SL(2, 27))$-equivalence. An embedding in $E_8(\mathbb{C})$ factors through $2E_7(\mathbb{C})$ (not $HSpin(16, \mathbb{C})$, as one can see by surveying characters of degree 16). Note that $SL(2, 27)$ does not embed in $3E_6(\mathbb{C})$ since in $3E_6(\mathbb{C})$ involution centralizers have types A_1A_5 and T_1D_5.

Rows for $A_1(29)$ in Table QE.

Let $L := PSL(2, 29)$. By Borel-Serre on the Borel subgroup of S, there is no projective embedding in $3E_6(\mathbb{C})$.

Serre's embedding of L in adjoint $E_7(\mathbb{C})$ [S96] lifts to $SL(2, 29)$ in $2E_7(\mathbb{C})$ by an eigenvalue argument [SP].

We show now that any S in $2E_7(\mathbb{C})$ is isomorphic to $SL(2, 29)$ and is Lie primitive. Let B be a Borel subgroup of S, isomorphic to $PSL(2, 29)$ or $SL(2, 29)$. Since B is supersolvable, it embeds in the normalizer of a torus. We deduce that $B/O_{29}(B)$ maps onto a cyclic subgroup of order 14 or 28 in the Weyl group. At this point, we see from the Weyl groups, that S is contained in no subgroup of type $G_2(\mathbb{C})$, $F_4(\mathbb{C})$ or $3E_6(\mathbb{C})$.

Since $W(E_7) \cong Sp(6,2) \times 2$, a Sylow 7-normalizer in S has the form $2 \times 7{:}6$. It follows that B contains an element x which corresponds to the -1 element of the Weyl group. In $2E_7(\mathbb{C})$, such x have order 4 and square to a generator of the center in $2E_7(\mathbb{C})$[GrElAb]. It follows that $|Z(B)| = 2$ and $S \cong SL(2,29)$.

From a $2B_7$ subgroup, we get existence of an embedding of L in $E_8(\mathbb{C})$.

If $S \cong SL(2,29)$ is embedded in $E_8(\mathbb{C})$, the central involution is of type $2A$, since otherwise, it has type $2B$, and so an embedding of S in $HSpin(16,\mathbb{C})$. The corresponding 16-dimensional orthogonal representation must correspond to a decomposition $16 = 1 + 15$ of irreducibles for $PSL(2,29)$. Then, the lift of this group to $Spin(16,\mathbb{C})$ is $PSL(2,29) \times 2$, a contradiction.

Rows for $A_1(q)$, $q \geq 31$, in Table QE.

For $q = 31, 32, 41, 49, 61$, the groups $PSL(2,q)$ have embeddings in $E_8(\mathbb{C})$ and no other exceptional group (see Section 3). For the number of embeddings, see [GR31], [GR41].

We show that the nonsimple groups $SL(2,q)$ does not embed in $E_8(\mathbb{C})$, for $q = 31, 41, 49, 61$. If S is such a subgroup of $E_8(\mathbb{C})$, it lies in the centralizer of an involution, t. If t has type $2B$, the 16-dimensional representation indicates $q \leq 33$, so the only case is $q = 31$. But such representations for S are not orthogonal. Suppose t has type $2A$. Then S has a degree 56 representation in which all constituents are faithful. For q odd, we have available degrees $(q \pm 1)/2, q-1, q$ and $q+1$. For $q = 41, 49$, and 61 we can not achieve such a degree 56 representation. For $q = 31$ and 32, the only projective embeddings are embeddings and the article [GR31] classifies them.

The group $PSL(2,37)$ does not embed in $E_8(\mathbb{C})$ [CG] and $SL(2,37)$ embeds in $E_8(\mathbb{C})$ only by a subgroup of type $2E_7(\mathbb{C})$. The number of classes of embeddings is 2 [KR].

Let $L := PSL(2,61)$. By Borel-Serre on the Borel subgroup of S, the only exceptional group in which S may embed is $E_8(\mathbb{C})$ and the group is $S \cong PSL(2,61)$.

In [CGL], it is shown that there is a unique class of subgroups of $E_8(\mathbb{C})$ isomorphic to $PSL(2,61)$. The associated character is rational, a sum of four degree 62 irreducibles. However, the number of classes of embeddings is two, both with the same character. Here is a proof. To embed L in $E_8(\mathbb{C})$, a subgroup J of the form $61{:}30$ is chosen (J is unique up to conjugacy), with element u of order 61, t of order 30 is chosen to satisfy $u^t = u^{46}$. An additional element w is sought, to satisfy $t^w = t^{-1}$, $(uw)^3 = 1$ and $wu^2w = t^{-1}u^{-2}wu^{30}$. It is shown [CGL] that there is a unique such w, given such u, t. The set of such pairs (u,t) in J lies in one orbit under $Aut(J)$. Every element of $Aut(J)$ is realized as the restriction of an automorphism of $Aut(L)$ to a Borel of L. It follows that the set of embeddings of L in $E_8(\mathbb{C})$ has two $E_8(\mathbb{C})$-conjugacy class and these two classes are fused by outer automorphisms of L.

Rows for $A_2(3)$ in Table QE.

For $PSL(3,3)$, we have a family of embeddings in $F_4(\mathbb{C})$ [CW97], 6.4, giving at least 2(1). Existence has been known a long time by the subgroup $3^3{:}SL(3,3)$ [Alek][CG87]. Note that there are no embeddings in $G_2(\mathbb{C})$ since a subgroup of the shape $AGL(2,3)$ does not embed (there is just one $G_2(\mathbb{C})$ class of elementary abelian groups of order 9 and if A is one of them, $N(A)/C(A) \cong Dih_{12}$).

Rows for $A_2(4)$ in Table QE.

Let $L := PSL(3,4)$. There are many possibilities for S, due to the large Schur multiplier, $4 \times 4 \times 3$, so this L requires a long discussion.

We shall prove that only the cases $|Z(S)| = 2, 4, 6$ can lead to subgroups of $E_8(\mathbb{C})$. The case $Z(S) = 1$ was eliminated by character restriction [CG].

Case: Suppose $3|\ |Z(S)|$.

Then S lies in a natural $3A_8$ or $3A_2E_6$ subgroup. The first case is eliminated since the only projective character of degree less than 10 and divisible by 3 is 6. In the second case, we claim that 6 divides $|Z(S)|$. We consider the possibility that $|Z(S)| = 3$. Then S is in the $3E_6$ factor, and on the 27-dimensional irreducible of $3E_6$, S has constituents of degrees 15 and 21 only, a contradiction. Thus, $|Z(S)|$ is even and so S is in an involution centralizer in $3E_6(\mathbb{C})$, a group of type $6A_1A_5$, and so S lies in a natural $6A_5$-subgroup and $|Z(S)| = 6$. There are just two such embeddings up to conjugacy, and these form an orbit under inverse-transpose in the normalizer of such a $6A_5$-subgroup or $Aut(S)$. We get entry 1(1) in Table QE.

There is no embedding of $6PSL(3,4)$ in $F_4(\mathbb{C})$ because the centralizer of an element x of order 3 in $F_4(\mathbb{C})$ has shape $3A_2^2$ when $x \in C(x)'$. This is impossible by the character table for $6PSL(3,4)$.

Case: Suppose $Z(S)$ contains an element f of order 4.

Since S is perfect, $f \in C_G(f)'$, where $G = E_8(\mathbb{C})$. This means that f is in class $4A, 4B, 4C$ or $4F$ with respective centralizer types A_7A_1, A_7T_1, A_3D_5 and D_7T_1. The irreducible projective degrees of S of at most 14 are 1, 6, 8 and 10; except for the trivial character, none of these allow a nontrivial invariant bilinear form.

In the latter two cases for $C_G(f)$, $\langle f \rangle$ is the center of the D_n-type component, and so only nontrivial irreducible constituents may occur in the $2n$-dimensional natural module. Since some self-dual combination of 6, 8 and 10s would have to add to 10 or 14, these last two possibilities do not occur. Therefore, one of the first two cases applies. The first may be eliminated as follows. The second component is a root A_1 subgroup whose G-centralizer is a $2E_7(\mathbb{C})$ subgroup. So, the other factor corresponds to an A_7-sublattice of the E_7-lattice. This means that the corresponding group has center which is cyclic of order 4 (rather than 8). However, this means that S lies in the A_7 factor and lifts in the simply connected group of type A_7 to a perfect central extension of $PSL(3,4)$ by a group of exponent divisible by 8, which contradicts the Schur multiplier theory for $PSL(3,4)$.

We conclude that the second case is the only one which occurs (and this type A_7 factor is simply connected), and we deduce that $Z(S) = \langle f \rangle$ and that the isomorphism type of S is determined (the other covering of L by a cyclic group of order 4 does not embed in $SL(8,\mathbb{C})$).

We show that any $4PSL(3,4)$ does not embed in $2E_7(\mathbb{C})$. If there were such an embedding, we would have an element f of order 4 in $2E_7(\mathbb{C})$ such that $f \in C(f)'$. A look at the class list of $2E_7(\mathbb{C})$ and the character table for covers of $PSL(3,4)$ shows no possibility. We mention the case that $C(f) \cong 4A_7$. If there were an embedding, it would lift to an embedding of a perfect group $8PSL(3,4)$ into $SL(8,\mathbb{C})$. There is no perfect group of shape $8PSL(3,4)$.

Case: $Z(S)$ contains a four-group.

Suppose that it does contain the four-group, V. There are four conjugacy classes of four groups in $E_8(\mathbb{C})$, according to the distribution of the classes $2A$ and $2B$ in them [CG]. If the distribution is AAA, $C(V) \cong T_2E_6{:}2$; if AAB, $C(V) \cong 2^2A_1^2D_6$; if ABB, $C(V) \cong T_1A_7$; and if BBB, $C(V) \cong 2^2D_4^2{:}2$. Since $V \leq S'' \leq C(V)''$, only AAB and BBB are possible [CG].

In the case AAB, there is an orthogonal representation of S with degree 12. The character table shows that the only representation of degree at most 12 preserving a bilinear form has two irreducible dual constituents of degrees 6 and 6. The image of S has shape $6{\cdot}PSL(3,4)$ and its lift to $Spin(12,\mathbb{C})$ is split, since the lift of the overgroup $GL(6,\mathbb{C})$ stabilizing the direct sum of maximal isotropics $6 \oplus 6$ is split. This forces $|Z(S)| = 6$, a contradiction.

Now we turn to the BBB case. Since the centralizer has components of types D_4^2, we are led to a sum of two degree 8 orthogonal representations. The character table shows that both must be irreducible and that on each S acts as $2PSL(3,4)$. To get an embedding of S in $HSpin(16,\mathbb{C})$, these two need to be realized as S modulo a pair of distinct subgroups of order 2 in $Z(S)$. If we take such an orthogonal representation of S, it lifts to an embedding of S in $HSpin(16,\mathbb{C})$ since on each component, the central involution has eigenvalues $1^8, -1^8$.

Case: $|Z(S)| = 2$.

We now demonstrate that the case $|Z(S)| = 2$ does occur. We use the degree 8 embedding of $4{\cdot}PSL(3,4)$ in $SL(8,\mathbb{C})$ and take a 2-fold quotient of this which occurs in a natural A_7-subgroup in $2E_7(\mathbb{C})$. This embeds S in $2E_7(\mathbb{C})$ so that $Z(S)$ is the center of $2E_7(\mathbb{C})$.

There is no embedding in $3E_6(\mathbb{C})$ or $E_6(\mathbb{C})$ because here the involution centralizers have shapes A_1A_5 and T_1D_5. Note that there is an embedding of S in $E_7(\mathbb{C})$, gotten from an embedding of $4PSL(3,4)$ in a natural A_7 subgroup of $2E_7(\mathbb{C})$. Any embedding of S in $E_7(\mathbb{C})$ must occur this way, due to the shapes of centralizers of noncentral elements of order 2 (A_1D_6) and of elements of order 4 which square to the central involution (A_7 and E_6T_1) plus the observation from Case $|Z(S)| = 4$ that if S lies in an A_7 factor, the A_7 factor is simply connected (the one which occurs in $2E_7(\mathbb{C})$ is not).

Rows for $A_2(5)$ in Table QE.

The group $PSL(3,5)$ embeds in $E_8(\mathbb{C})$ by existence of the subgroup $5^3{:}SL(3,5)$ [Alek][GR87]. Its character is a sum of a degree 124 irreducible and its complex conjugate, the unique such pair with trace -4 on the involutions and trace -2 on the elements of order 3 [CG]. We know of no conjugacy results on embeddings.

The character table rules out embeddings in $2E_7(\mathbb{C})$ since a degree 56 character would have the principal irreducible with multiplicity at least 25, forcing involutions to have too large a trace.

Rows for ${}^2A_2(3)$ in Table QE.

This group embeds in $G_2(\mathbb{C})$ [CW83][GrG2] hence in all exceptional groups.

Rows for ${}^2A_2(8)$ in Table QE.

Any projective embedding of $PSU(3,8)$ into $E_8(\mathbb{C})$ factors through an embedding of $PSU(3,8)$ into $2E_7(\mathbb{C})$ [CG]. The existence and classifications of the latter embeddings was done in [GRU].

Rows for ${}^2A_3(2)$ in Table QE.

Let $L := PSU(4,2)$. The group $2{\cdot}PSU(4,2) \cong Sp(4,3)$ does embed in $F_4(\mathbb{C})$ via $4A_3 \leq 2^2D_4$. It does not embed in $G_2(\mathbb{C})$ since the centralizer of an involution in G_2 has shape $2A_1A_1$.

The simple group does not embed in $F_4(\mathbb{C})$, originally proved in [CW97] by character theory and the classification of orbits on the 26-dimensional module. Here is a new proof. If L does embed, look at the parabolic P of the form $2^4{:}\,Alt_5$. Since 5 does not divide the order of the Weyl group of F_4, $O_2(P) \cong 2^4$ is nontoral. But then we contradict the structure of the maximal nontoral elementary abelian 2-subgroups and their normalizers in $F_4(\mathbb{C})$[GrElAb], (1.8). The simple group does embed in $3E_6(\mathbb{C})$ due to its 6-dimensional representation and the subgroup $6A_5$ of $3E_6(\mathbb{C})$.

Rows for $^2A_3(3)$ in Table QE.

Let $L := PSU(4,3)$. This group has a large multiplier, $3 \times 3 \times 4$.

To begin with, consider the case where 9 divides the order of $Z(S)$. Let V be the group of order 9 in $Z(S)$, a "nine-group". Then each element of order 3 is in the commutator subgroup of its centralizer, whence the centralizer has type A_8 or A_2E_6, and clearly the former is impossible since then L would have a perfect extension by a cyclic group of order 9. So, S is in the $3E_6(\mathbb{C})$-factor. A study of elements of order 3 in $3E_6(\mathbb{C})$, using $V \leq C(V)'$, shows that $C(V) \cong 3^2A_2^4$, whence V is $3B$-pure [GrElAb]. But this group contains no quasisimple finite group with central quotient $PSU(4,3)$, a contradiction.

We imitate the discussion for $PSL(3,4)$. Exactly the same argument shows that $Z(S)$ has order divisible by 6 if its order is divisible by 3 and that S is a unique extension of L by a cyclic group of order 6 and S is embedded in a natural $6A_5$-subgroup. A consequence is another proof that it does not have order divisible by 9.

Suppose now that $Z(S)$ has order divisible by 4; let f be an element of order 4. We now follow the corresponding discussion in the above $PSL(3,4)$-case and now get a contradiction since the only nontrivial irreducible degree at most 14 is 6.

Next, we eliminate the case that $Z(S)$ has order 2. Their character tables show that L and S do not embed in $GL(m,\mathbb{C})$, for $m \leq 19$. So, the central involution does not have type $2B$, whence S embeds in a group of type $2E_7(\mathbb{C})$. The character table implies that there are fixed points on a degree 133 module, whence the embedding is not Lie primitive. Since low degree classical overgroups are impossible, it follows that S lies in a $3E_6(\mathbb{C})$ subgroup, whence in an even smaller subgroup (of classical type) since it has a central involution, a final contradiction.

Finally, $Z(S) = 1$ is out since $PSL(3,4)$ does not embed in $E_8(\mathbb{C})$ and $PSL(3,4)$ is in $PSU(4,3)$.

We conclude that $|Z(S)| = 6$ and S is the particular 6-fold covering which embeds in $SL(6,\mathbb{C})$. We deduce an embedding in $3E_6(\mathbb{C})$. If there were an embedding in $F_4(\mathbb{C})$, since the element of order 3 in the center is in the commutator group of its centralizer, we deduce an embedding of S in a $3A_2A_2$ subgroup, which is impossible by the character table of S.

Rows for $D_4(2)$ in Table QE.

In [CG], it was shown that every central extension of $\Omega^+(8,2)$ and of $Sp(6,2)$ and of $\Omega^{\pm}(6,2)$ embeds in $E_8(\mathbb{C})$.

We now show that $\Omega^+(8,2)$ embeds in $2E_7(\mathbb{C})$. In W_{E_8}, we have a copy of $2{\cdot}Sp(6,2)$ and $Sp(6,2)$. Let $\langle E, t\rangle$ be as in (4.5) with $t \in 2A$. From $W_{E_8} \hookrightarrow SO(8,\mathbb{C}) \to PSO(8,\mathbb{C})$, and $W_{E_8}/\{\pm 1\} \cong \Omega^+(8,2)$, we get an embedding of $\Omega^+(8,2)$ and at least one conjugacy class of embeddings of $Sp(6,2)$ in $C(\langle E, t\rangle) \le C(t) \cong 2A_1E_7$ and the images lie in the $2E_7$-factor. (It seems possible that the two nonconjugate subgroups of $PSO(8,\mathbb{C})$ are conjugate in $C(t)$.)

There is no projective embedding in $G_2(\mathbb{C})$ [CW83][GrG2].

The full covering group lies in $Spin(8,\mathbb{C})$, hence embeds in all exceptional groups containing $F_4(\mathbb{C})$.

We claim that $S \cong 2{\cdot}D_4(2)$ does not embed in $2E_7(\mathbb{C})$. Suppose otherwise. Then the restriction of the degree 56 irreducible must be a sum of faithful irreducibles (otherwise, S embeds in the centralizer of an involution in $2E_7(\mathbb{C})$, a group of type A_1D_6, impossible). Such decompositions involve only the irreducibles of degrees 8 and 56. Then the $3A$ class of S has trace 40 or 11, neither of which are allowed in $2E_7(\mathbb{C})$.

We claim that L does not embed in $3E_6(\mathbb{C})$. This is obvious since nontrivial character degrees start at 28. Alternatively, one can quote the nonembedding of $Sp(6,2)$ in $3E_6(\mathbb{C})$ (below).

Rows for $PSp(4,5)$ in Table QE.

Procedure XR shows that there are no E_8-feasible partial characters of $PSp(4,5)$ on the classes 1A, 2A, 2B, 3A, 3B (we need to use thirteen partial irreducible characters of degrees 1, 13, 40, 65, 65, 78, 90, 104, 104, 104, 130, 156, and 208 in this analysis).

We now discuss the central extension $S \cong Sp(4,5) \cong 2PSp(4,5)$. Since this group has faithful irreducibles of dimensions only 12, 52 and higher, it does not embed as a subgroup of $2E_7(\mathbb{C})$ with a common center due to this group's faithful irreducible of degree 56. So, if it is in $2E_7(\mathbb{C})$, it must lie in the centralizer of a noncentral involution, hence a group of type A_1D_6 This is a contradiction since $Sp(4,5)$ has no nontrivial orthogonal representation of degree 12. There is one of degree 13, and so we get an embedding of $Sp(4,5)$ into $E_8(\mathbb{C})$ via a natural $Spin(13,\mathbb{C})$-subgroup of $HSpin(16,\mathbb{C})$.

Rows for $B_3(2) = C_3(2) = Sp(6,2)$ in Table QE.

We see that $Sp(6,2)$ embeds in $2E_7(\mathbb{C})$ because $PSO(8,2)$ does. The covering group $2Sp(6,2)$ is contained in a natural 2^2D_4 subgroup of $F_4(\mathbb{C})$, hence in overgroups. The covering group does not embed in $G_2(\mathbb{C})$ since centralizers of involutions in $G_2(\mathbb{C})$ have shape $2A_1A_1$.

Here is a proof that $Sp(6,2)$ does not embed in $3E_6(\mathbb{C})$ (this implies nonembedding in $F_4(\mathbb{C})$ and $G_2(\mathbb{C})$). From [CW97], we see that on the 27-dimensional module for $3E_6(\mathbb{C})$, involutions have trace 3 and -5. In $Sp(6,2)$, the class designated $3C$ [Atlas] has only positive traces on irreducibles of degree at most 27, so must have trace 3 on the 27-dimensional module in case there is an embedding. There is no way to make a character of degree 27 which has trace 3 on this class, whence no embedding exists.

Rows for $Sz(8)$ in Table QE. This was treated in [GR8], which proves that there are exactly three conjugacy classes of embeddings in $E_8(\mathbb{C})$, forming an orbit under $Aut(S)$. All are Lie primitive. See [GRU] for a classification of embeddings of $Sz(8)$ in groups of type E_7 over fields of characteristic 5.

Rows for $G_2(3)$ in Table QE.

Procedure XR shows that there are no E_7-feasible partial characters on the classes 1A, 2A, 3A, 3C, 3E (we need to use eight partial characters of degrees 1, 14, 64, 78, 91, 91, 91, and 104 in this analysis). For a machine free argument, we observe that if $G_2(3)$ did embed in $2E_7(\mathbb{C})$, the trace of any element of order 3 in $G_2(3)$ on the 56-dimensional module M is in $\{-7,-25,2,20\}$ [CG].

There is just one nontrivial irreducible character for $G_2(3)$ with degree at most 56; call it χ. Then $\chi(1)=14$, and χ has values 5, 5, -4, 2, and -1 on the elements of order 3 in $G_2(3)$. If the 56-dimensional representation of $2E_7(\mathbb{C})$ restricted to $G_2(3)$ to give a copies of the trivial character and b copies of χ, we would have $56 = a + 14b$, for an integer b with $1 \leq b \leq 4$. Note that $a \equiv 0 \pmod{14}$. Since M has a nondegenerate alternating form, we have $b \in \{2,4\}$. Taking $x \in G_2(3)$ with $|x| = 3$, and $\chi(x) = -4$, we get $a - 4b \in \{-7,-25,2,20\}$. Since $a - 4b$ is even, we have $a - 4b \in \{2,20\}$, whence $a \geq 1$ and $b \leq 3$. We deduce that $b = 2$ and $a = 28$. Now take $y \in G_2(3)$, $|y| = 3$, and $\chi(y) = 2$. We have $28 + 2\chi(y) = 28 + 4 = 32 \in \{-7,-25,2,20\}$, a final contradiction.

The simple group $G_2(3)$ embeds in $E_8(\mathbb{C})$ by a D_7 subgroup.

We discuss the nontrivial central extension which could occur here, namely $S \cong 3{\cdot}G_2(3)$. From [GrSM1] and the $3B$-criterion (3.1.h), a central element z of order 3 is in $3B$, whence S is in a natural $3E_6(\mathbb{C})$ subgroup. Now, using Procedure XR with the $G_2(3)$ classes 1A, 2A, 3A, 3C, and 3E we find that the only E_6-feasible characters of L are either irreducible of degree 78 or the sum of irreducible characters of degrees 14 and 64. However, the tensor product of either of these potential E_6-feasible characters with the 27-dimensional irreducible character of the triple cover of L decomposes as a sum of two irreducible characters of degrees

1728 and 378. However the 27-dimensional representation of $3E_6(\mathbb{C})$ is a module for the corresponding Lie algebra, and therefore it appears as a constituent in the tensor product of $3E_6(\mathbb{C})$-modules of degrees 27 and 78. It follows that S does not embed in $3E_6(\mathbb{C})$.

Rows for $^3D_4(2)$ in Table QE.

There is an embedding in $F_4(\mathbb{C})$. Any embedding in $E_8(\mathbb{C})$ has a uniquely determined E_8-adjoint character $14.1_a + 7.26_a + 52_a$ [CG]. We are not aware of any conjugacy result. Note that it does not embed in $G_2(\mathbb{C})$ or else its centralizer would have dimension at least 52, due to the subgroup of $E_8(\mathbb{C})$ of type $F_4(\mathbb{C}) \times G_2(\mathbb{C})$, contradicting the character statement. Since S contains an extraspecial subgroup 2^{1+8} (in a maximal parabolic), it is clear that any faithful representation has degree at least 16, so there is no embedding of S in $G_2(\mathbb{C})$.

Rows for $^2F_4(2)'$ in Table QE.

Note that $^2F_4(2)'$ does not embed in $F_4(\mathbb{C})$ since every semisimple element of $F_4(\mathbb{C})$ is real, whereas elements of order 16 in $^2F_4(2)'$ are nonreal. There is an embedding in $3E_6(\mathbb{C})$ with a uniquely determined E_8-adjoint character $8.1_a + 3.27_c + 3.27_d + 78_a$ [CG][CW97]. We are not aware of any conjugacy result.

Rows for M_{11} in Table QE.

This group occurs in a natural D_5 subgroup, so is in $3E_6(\mathbb{C})$. If it were embedded in $F_4(\mathbb{C})$, the subgroup $PSL(2,11)$ would be embedded, a contradiction.

Rows for M_{12} in Table QE.

Procedure XR shows that there are two E_8-feasible partial characters of M_{12} on the classes 1A, 2A, 2B, 3A, 3B (we need to use thirteen partial irreducible characters of degrees 1, 11, 16, 45, 54, 55, 55, 66, 99, 120, 144, and 176 in this analysis). Each of these two partial characters has several decompositions as a nonnegative integral combination of partial irreducible characters, and we must consider all of the corresponding combinations of irreducible characters.

The possible decompositions of the first feasible partial character are made up from two 11-dimensional irreducibles, a 16-dimensional irreducible and either the sum $1+55_a+55_a+99$ or the sum $45+55_a+55_a+55_a$. In any case, the decomposition has a single nonreal character, of degree 16. It follows that this partial character cannot come from an embedding into $E_8(\mathbb{C})$.

The possible decompositions of the second feasible partial character are either $1+1+1+1+1+1+45+45+120$ together with two 16-dimensional irreducibles, or $1+45+45+45+45+45$ together with two 11-dimensional irreducibles. In the first of these cases, the corresponding character would have value 8 on the class 5A of M_{12}, but no element of order 5 in $E_8(\mathbb{C})$ has this trace on the natural module. In the second case, the corresponding character would have value -6 on the class 6A of M_{12}, but no element of order 6 in $E_8(\mathbb{C})$ has this trace on the natural module. It follows that there is no embedding of the simple group M_{12} into $E_8(\mathbb{C})$.

We prove that $S \cong 2{\cdot}M_{12}$ has an embedding in $E_8(\mathbb{C})$ and it is unique up to conjugacy if it centralizes an element of class $2B$ and in that case the central involution of S is in $2A$.

There is a degree 12 orthogonal representation which gives an embedding of $S \cong 2{\cdot}M_{12}$ in $Spin(12, \mathbb{C})$, which lies in $2E_7(\mathbb{C})$ since the root lattice of D_6 is a direct summand of the root lattice of E_7 [GrElAb].

Suppose that S centralizes some t in $2B$. Then S lies in a natural D_6 subgroup of $C(t)$, and we have a unique character of degree 12 and so we have S unique up to conjugacy in $E_8(\mathbb{C})$. In particular, this forces the central involution of $S \cong 2{\cdot}M_{12}$ to be in $2A$ since the fixed point free involutions of M_{12} on the degree 12 representation have spectrum $1^6, -1^6$ so lift to elements of order 4 in the spin group [GrElAb].

We prove that $2M_{12}$ is not in $3E_6(\mathbb{C})$. If so, it is in the centralizer of an involution, which has type D_5T_1 or A_1A_5. The character table of $2M_{12}$ allows no non-trivial irreducible representations of degree at most 9 and just one of degree 10, which is not orthogonal, contradiction.

Rows for HJ in Table QE.

Procedure XR shows that there are no E_8-feasible partial characters of HJ on the classes 1A, 2A, 2B, 3A, 3B (we need to use thirteen partial irreducible characters of degrees 1, 14, 21, 36, 63, 70, 90, 126, 160, 175, 189, 224, and 225 in this analysis).

Since $6A_5$ is in $3E_6(\mathbb{C})$, we get an embedding of the double cover $2HJ$ in $3E_6(\mathbb{C})$ and overgroups. Since $2HJ$ has an embedding in $Sp(6, \mathbb{C})$, we deduce an embedding in $F_4(\mathbb{C})$ because of the involution centralizer of shape $2A_1C_3$ in $F_4(\mathbb{C})$. In fact, we can classify embeddings in $F_4(\mathbb{C})$ up to conjugacy.

An embedding of $2HJ$ in $F_4(\mathbb{C})$ factors through an embedding in an involution centralizer. In $F_4(\mathbb{C})$ an involution centralizer is isomorphic to $Spin_9(\mathbb{C})$ or $2A_1C_3$, and the character table excludes the former. There are two degree 6 characters associated to embeddings in $Sp(6, 2)$. By (4.2.i), we deduce that $2HJ$ has exactly two conjugacy classes of embeddings in $F_4(\mathbb{C})$.

NEED DEFINITIONS FOR B, J, T, A, P (BOOK, JOURNAL ARTICL ARTICLE IN BOOK, PERSONAL COMMUNICATION.) STUDY JGT STYLE PAGE.

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