MATH 451: Exam I (Solution)
Winter 2019

NAME: ______________________________

Read all questions carefully.
There are **four** problems.
Show all your work. No work, no points.
No book, no notes, no calculators, no electronics.

Problem	**Points Possible**	**Points Earned**
1	25	
2	25	
3	25	
4	25	
Total	100	

1. (25 points) (a)Prove that for any real numbers a, b,

$$\big||a| - |b|\big| \le |a - b|.$$

Proof: By the triangle inequality, we have for any real numbers a and b,

$$|a + b| \le |a| + |b|.$$

So

$$|a| = |a - b + b| \le |a - b| + |b|,$$

and this gives

$$|a| - |b| \le |a - b|. \tag{1}$$

Replacing a by b, and b by a in (1) gives

$$|b| - |a| \le |b - a| = |a - b|. \tag{2}$$

Since

$$\big||a| - |b|\big| = \begin{cases} |a| - |b|, & \text{if } |a| \ge |b| \\ |b| - |a|, & \text{if } |b| \ge |a| \end{cases} \tag{3}$$

we have, by (1),(2) and (3),

$$\big||a| - |b|\big| \le |a - b|.$$

(b) Prove that for any real numbers a, b,

$$|a| - |b| \le |a + b|.$$

Proof: by the triangle inequality, we have

$$|a| = |a + b - b| \le |a + b| + |-b| = |a + b| + |b|$$

so

$$|a| - |b| \le |a + b|.$$

2. (25 points) (a) Write the definition for the sequence $(a_n)_{n\in\mathbb{N}}$ to converge to a as n goes to ∞.

We say a sequence (a_n) converges to a if for any $\epsilon > 0$, there is $N \in \mathbb{N}$, such that for all $n > N$,

$$|a_n - a| < \epsilon.$$

(b) Let $a_n = \sqrt{n^2 - n} - n$. Does the sequence $(a_n)_{n\in\mathbb{N}}$ converge? Prove your assertion.

We have

$$a_n = \sqrt{n^2-n} - n = (\sqrt{n^2-n} - n)\frac{\sqrt{n^2-n}+n}{\sqrt{n^2-n}+n} = \frac{-n}{\sqrt{n^2-n}+n} = \frac{-1}{\sqrt{1-\frac{1}{n}}+1}.$$

Now we want to show $\lim_{n\to\infty}\sqrt{1-\frac{1}{n}} = 1$. We know

$$1 - \frac{1}{n} \leq \sqrt{1-\frac{1}{n}} \leq 1 \qquad \text{for all } n \in \mathbb{N}.$$

so by the Squeezing Lemma, we have, because $\lim_{n\to\infty}(1-\frac{1}{n}) = 1$, and $\lim_{n\to\infty} 1 = 1$, $\lim_{n\to\infty}\sqrt{1-\frac{1}{n}} = 1$. Now the limit Theorem implies that the sequence (a_n) converges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{-1}{\sqrt{1-\frac{1}{n}}+1} = \frac{-1}{\lim_{n\to\infty}\sqrt{1-\frac{1}{n}}+1} = -\frac{1}{2}.$$

(c) Let

$$b_n = 2^{(-1)^n}.$$

(c1)Write out the first 4 terms, b_1, b_2, b_3 and b_4, of the sequence.

$b_1 = 1/2$, $b_2 = 2$, $b_3 = 1/2$, $b_4 = 2$.

(c2) Does the sequence $(b_n)_{n\in\mathbb{N}}$ converge? Prove your assertion.

No. We know $b_n = 1/2$ for n odd and $b_n = 2$ for n even. If (b_n) converges, say to b, then for $\epsilon = 1/2 > 0$, there is a $N \in \mathbb{N}$, such that for all $n > N$, $|b_n - b| < 1/2$. Then for all odd numbers $n > N$, $|1/2 - b| < 1/2$, and for all even numbers $n > N$, $|2 - b| < 1/2$. This gives, by triangle inequality,

$$3/2 = |2 - 1/2| = |2 - b + b - 1/2| \leq |2 - b| + |b - 1/2| < 1/2 + 1/2 = 1.$$

This is a contradiction. So (b_n) does not converge.

3. Assume that the sequence $(a_n)_{n\in\mathbb{N}}$ diverges to $+\infty$. Show that $(a_n)_{n\in\mathbb{N}}$ does not converge.

Proof. Since $(a_n)_{n\in\mathbb{N}}$ diverges to $+\infty$, so for any $M \in \mathbb{R}$, there is $N \in \mathbb{N}$, such that for all $n > N$,

$$a_n > M.$$

This implies that (a_n) is not bounded.

Since we know any convergence sequence is bounded, so (a_n) does not converge.

4. Assume that $(a_n)_{n\in\mathbb{N}}$ is a bounded and monotone sequence. Show that $(a_n)_{n\in\mathbb{N}}$ converges.

Proof: Case 1. Assume that $(a_n)_{n\in\mathbb{N}}$ is increasing. That is, for all $n \in \mathbb{N}$,

$$a_{n+1} \geq a_n.$$

Because (a_n) is bounded, there is $M > 0$, such that

$$-M < a_n < M, \qquad \text{for all } n \in \mathbb{N}.$$

And by the Completeness Axiom, the least upper bound, $\sup\{a_n | \forall n \in \mathbb{N}\}$, exist in $\mathbb{R}$. Write

$$a := \sup\{a_n | \forall n \in \mathbb{N}\}.$$

Want to show a_n converges to a as n tends to ∞. First we have

$$a_n \leq a \qquad \text{for all } n. \tag{4}$$

Now since a is the least upper bound, so for any $\epsilon > 0$, $a - \epsilon$ is not an upper bound. So there is $N \in \mathbb{N}$, such that $a_N > a - \epsilon$. Now since (a_n) is increasing, so for all $n > N$,

$$a_n \geq a_N > a - \epsilon. \tag{5}$$

Combining (4) and (5), we have shown for all $n > N$,

$$a - \epsilon < a_n \leq a < a + \epsilon.$$

Hence (a_n) converges to a.

Case 2: Assume that (a_n) is decreasing, then the sequence $(-a_n)$ is increasing, and since $(-a_n)$ is also bounded, the above argument in case 1 shows that $(-a_n)$ converges. By the limit Theorem, we also have (a_n) converges.