MATH 451: Exam II
Winter 2019

NAME: ______________________________

Read all questions carefully.
There are **four** problems, plus a bonus problem.
You should make sure that you have finished the main part of the exam before working on the bonus problem.
Show all your work. No work, no points.
No book, no notes, no calculators, no electronics.

Problem	Points Possible	Points Earned
1	25	
2	25	
3	25	
4	25	
Total	100	
Bonus	25	

1. (25 points) a) Consider the sequence

$$\left(1, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \ldots, \frac{1}{n}, 1 - \frac{1}{n}, \ldots\right)$$

Find its set of subsequential limits. What are its lim sup and lim inf ?

Solution: Observe that there are exactly two converging subsequences: $s_{2n-1} = \frac{1}{n}$, $s_{2n} = 1 - \frac{1}{n}$. So the set of sub-sequential limits is $S = \{1, 0\}$.

$$\limsup s_n = 1, \qquad \liminf s_n = 0.$$

b) Assume that there is a real number b and a natural number N_0, such that for all $n > N_0$, we have $s_n \leq b$. Show that $\limsup\, s_n \leq b$.

Proof: Let $v_N = \sup\{s_n \,|\, n > N\}$. By definition, we know

$$\limsup\, s_n = \lim_{N \to \infty} v_N.$$

By the assumption, we have

$$v_{N_0} = \sup\{s_n \,|\, n > N_0\} \leq b.$$

Since v_N is a decreasing sequence, so for all $N \geq N_0$, $v_N \leq b$. Let $N \to \infty$, we get

$$\lim_{N \to \infty} v_N \leq b.$$

This proves that $\limsup\, s_n \leq b$.

2. (25 points) True or false? If the statement is true, give a brief argument to justify your answer. If it is false, give a counterexample.

(a) Assume that for all $n \in \mathbb{N}$, $a_n < b_n$, and $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} b_n$ exist. Then $\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n$.

False. Let

$$a_n = 0, \qquad b_n = \frac{1}{n} \qquad \text{for all } \ n \in \mathbb{N}.$$

Then

$$a_n < b_n \qquad \text{for all } \ n,$$

but

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0.$$

(b) If $\lim_{n\to\infty} x_n y_n = 0$, then either $\lim_{n\to\infty} x_n = 0$ or $\lim_{n\to\infty} y_n = 0$.

False. Let

$$x_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases} \qquad y_n = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Then $x_n y_n = 0$ for all $n \in \mathbb{N}$, so

$$\lim_{n\to\infty} x_n y_n = 0,$$

but $\lim x_n$, $\lim y_n$ do not exist.

3. (25 points) (a) Let $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Is the sequence $(s_n)_{n\in\mathbb{N}}$ Cauchy? Prove your assertion.

Yes. Let $n > m$. We know $\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$, for $k > 1$, so

$$0 < s_n - s_m = \sum_{k=m+1}^{n} \frac{1}{k^2} \leq \sum_{k=m+1}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{m} - \frac{1}{n} < \frac{1}{m}.$$

Therefore

$$|s_n - s_m| \leq \frac{1}{m}.$$

Let $\epsilon > 0$, and $N = \frac{1}{\epsilon}$. We have, for any $n > m > N$,

$$|s_n - s_m| \leq \frac{1}{m} < \epsilon.$$

This shows that (s_n) is a Cauchy sequence.

(b) Consider the series $\sum_{n=2}^{\infty} \frac{n+1}{n^3-2}$. Is it convergent? Prove your assertion.

Yes. Observe that

$$0 < \frac{n+1}{n^3-2} \leq \frac{n+n}{\frac{1}{2}n^3} = \frac{4}{n^2}, \qquad \text{for } n \geq 2.$$

Since $\sum \frac{1}{n^2}$ is convergent, by the comparison test, we know $\sum_{n=2}^{\infty} \frac{n+1}{n^3-2}$ is also convergent.

4. (25 points) Assume that $(s_n)_{n\in\mathbb{N}}$ is a bounded sequence. Give a step by step construction of a converging subsequence of $(s_n)_{n\in\mathbb{N}}$.

Assume that for all $n \in \mathbb{N}$, $s_n \in [a, b]$ for some $a, b \in \mathbb{R}$.

Step 1: subdivide the interval $[a, b]$ by the midpoint $m = \frac{a+b}{2}$ into 2 equal length subintervals $[a, m]$ and $[m, b]$. Since there are infinitely many $s_n{}'s$ in $[a, b]$, at least one of the subintervals $[a, m]$ or $[m, b]$ contains infinitely many $s_n{}'s$. We choose one of the subintervals containing infinite many $s_n{}'s$ and name it $[a_1, b_1]$. Continue this process, we get a sequence of subintervals $[a_k, b_k]$, $k = 1, 2, \ldots, n \ldots$, with the properties that each $[a_k, b_k]$ contains infinite many $s_n{}'s$, and

$$[a_{k+1}, b_{k+1}] \subset [a_k, b_k], \qquad \text{for } k \in \mathbb{N}, \tag{1}$$

$$b_k - a_k = \frac{1}{2}(b_{k-1} - a_{k-1}) = \cdots = \frac{1}{2^k}(b - a), \qquad \text{for } k \in \mathbb{N}. \tag{2}$$

Step 2: Since there are infinitely many $s_n{}'s$ in $[a_1, b_1]$, we choose an arbitrary $s_{n_1} \in [a_1, b_1]$. Assume that we have chosen $s_{n_1}, \ldots s_{n_k}$, such that $s_{n_i} \in [a_i, b_i]$, and $n_1 < n_2 \cdots < n_k$. Since there are infinitely many $s_n{}'s$ in $[a_{k+1}, b_{k+1}]$, and there are at most n_k terms of $s_n{}'s$ before the term s_{n_k}, we can choose $s_{n_{k+1}} \in [a_{k+1}, b_{k+1}]$, such that $n_{k+1} > n_k$.

Continue this process for $k = 1, 2, \ldots, n \ldots$, we obtain a subsequence $(s_{n_k})_{k\in\mathbb{N}}$.

Step 3: We know $a_k \le s_{n_k} \le b_k$ for $k = 1, 2, \ldots, n \ldots$. By (1), for all $m, k \ge N$, $s_{n_k}, s_{n_m} \in [a_N, b_N]$; and by (2),

$$|s_{n_k} - s_{n_m}| \le b_N - a_N = \frac{1}{2^N}(b - a).$$

Therefore (s_{n_k}) is a Cauchy sequence,[1] hence it converges. This gives a converging subsequence of (s_n).

Remark: The constructional proof for the existence of a subsequence converging to $\limsup s_n$, or the constructional proof for the existence of a monotonic subsequence will also be valid here.

[1]For any $\epsilon > 0$, there is $N \ge \frac{b-a}{\epsilon}$, such that for all $k, m \ge N$, $|s_{n_k} - s_{n_m}| \le b_N - a_N = \frac{1}{2^N}(b - a) \le \frac{1}{N}(b - a) \le \epsilon$.

5. Bonus. (25 points) Let $a_n, b_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, and assume that the intervals $[a_n, b_n]$ satisfy
$$[a_{n+1}, b_{n+1}] \subset [a_n, b_n], \qquad \text{for all } n \in \mathbb{N}.$$
Assume further that
$$\lim_{n\to\infty} (b_n - a_n) = 0.$$
Show that there is exactly one real number c, satisfying
$$c \in [a_n, b_n], \qquad \text{for all } n \in \mathbb{N}.$$
(Hint: plot the intervals $[a_n, b_n]$ on the real line, and think what could be the real number c that is in all the intervals $[a_n, b_n]$.)

Proof. Because
$$[a_{n+1}, b_{n+1}] \subset [a_n, b_n], \qquad \text{for all } n,$$
we have
$$a_{n+1} \geq a_n, \quad b_{n+1} \leq b_n, \qquad \forall n,$$
so (a_n) is an increasing sequence, (b_n) is a decreasing sequence, and both sequences are bounded below by a_1 and bounded above by b_1. So (a_n) and (b_n) are convergent sequences. By the assumption that
$$\lim_{n\to\infty} (b_n - a_n) = 0,$$
we have
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$
Let $c = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Now since (a_n) is increasing, we know $\lim a_n = \sup\{a_n \,|\, n \in \mathbb{N}\}$, and since (b_n) is decreasing, we know $\lim b_n = \inf\{a_n \,|\, n \in \mathbb{N}\}$. Therefore we have
$$a_n \leq c \leq b_n, \qquad \text{for all } n.$$
This shows that there is at least one real number c, such that
$$c \in [a_n, b_n], \qquad \text{for all } n \in \mathbb{N}.$$

Now we want to show that there is no more than one real number c satisfying $a_n \leq c \leq b_n$ for all n. If not, assume that there are c_1 and c_2, satisfying
$$c_1, c_2 \in [a_n, b_n] \qquad \text{for all } n$$
then
$$|c_1 - c_2| \leq |b_n - a_n| \qquad \text{for all } n$$
Because
$$\lim_{n\to\infty} (b_n - a_n) = 0,$$
we have $|c_1 - c_2| = 0$. This implies $c_1 = c_2$. So there can be no more than one real number c satisfying $c \in [a_n, b_n]$ for all n.