MATH 451 HOMOWORK SET 13 (ADDITIONAL)

1. Assume that the function f is continuous on $[a, b]$, and differentiable on (a, b), where $a, b \in \mathbb{R}$. Assume further that

$$\lim_{x \to a+} f'(x) = L.$$

Show that the one sided derivative

$$f'_+(a) := \lim_{x \to a+} \frac{f(x) - f(a)}{x - a}$$

also exist and

$$f'_+(a) = L.$$

2. Let I be an open interval containing x_0, and f is twice differentiable on I. Show that

$$\lim_{h \to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} = f''(x_0).$$

3. Show that if $0 < x \le 1$, then the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$ converges, and we have

$$\ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k.$$

4. Show that

$$1 + \frac{x}{3} - \frac{x^2}{9} < (1 + x)^{1/3} < 1 + \frac{x}{3}, \quad \text{for } x > 0.$$

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is n-times differentiable, and $f^{(n)}$ is continuous on $\mathbb{R}$. Show that $f^{(k)}(0) = 0$, for $k = 0, \ldots n$, if and only if

$$\lim_{x \to 0} \frac{f(x)}{x^n} = 0.$$

6. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has derivatives of all orders, and

$$\begin{cases} f''(x) - f'(x) - f(x) = 0, & \text{for all } x \\ f(0) = 0, \ f'(0) = 2 \end{cases} \tag{0.1}$$

Find a recursive formula for the coefficients of the Taylor series for f at $x = 0$. Show that the Taylor series converges for every $x \in \mathbb{R}$.