PARAMETERIZING CONJUGACY CLASSES OF MAXIMAL UNRAMIFIED TORI 
VIA BRUHAT-TITS THEORY

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ABSTRACT. Let \( k \) denote a field with nontrivial discrete valuation. We assume that \( k \) is complete with perfect residue field \( f \). Let \( G \) denote the group of \( k \)-rational points of a reductive, linear algebraic group \( G \) defined over \( k \). A subgroup in \( G \) is said to be unramified if it is a connected reductive subgroup of \( G \) whose reduced Bruhat-Tits building contains a hyperspecial vertex. Let \( \mathcal{C} \) denote the set of \( G \)-conjugacy classes of pairs \((H, x)\) where \( H \) is a maximal rank unramified subgroup of \( G \) and \( x \) is a hyperspecial point in the reduced Bruhat-Tits building of \( H \). Let \( I^c \) denote the set of pairs \((F, H)\) where \( F \) is a facet in the Bruhat-Tits building of \( G \) and \( H \) is a \( f \)-cuspidal maximal rank connected reductive subgroup in \( G_F \) (the connected reductive \( f \)-group associated to \( F \)). There is a natural equivalence relation, to be denoted \( \sim \), on \( I^c \). We show that there is a bijective correspondence between the set \( I^c/\sim \) and \( \mathcal{C} \). From this, we derive a classification of the conjugacy classes of maximal unramified tori in \( G \) and, when \( f \) is finite and \( G \) is unramified, we determine which of these conjugacy classes of tori are stably conjugate.

0. INTRODUCTION

One of the main results of this paper is a uniform parametrization of the conjugacy classes of maximal unramified tori in a reductive \( p \)-adic group. This classification matches conjugacy classes of maximal unramified tori in \( G \) with certain equivalence classes that arise naturally from Bruhat-Tits theory. The motivation for this result comes from harmonic analysis; specifically, from J.-L. Waldspurger’s papers [13, 14]. Using the parametrization scheme discussed in this paper, D. Kazhdan and I [6] have been able to generalize, in a uniform manner, some of the results in [13].

We now discuss the contents of this paper.

Let \( k \) denote a field with nontrivial discrete valuation. We assume that \( k \) is complete with perfect residue field \( f \). Let \( G \) denote the group of \( k \)-rational points of a reductive, linear algebraic group \( G \) defined over \( k \). Let \( G^\circ \) denote the group of \( k \)-rational points of the identity component \( G^\circ \) of \( G \). Let \( \mathcal{B}(G) \) denote the Bruhat-Tits building of \( G^\circ \). We let \( \mathcal{B}^\text{red}(G) \) denote the reduced Bruhat-Tits building of \( G^\circ \).

A subgroup in \( G \) is called unramified if it is a connected reductive \( k \)-subgroup of \( G \) whose reduced building contains a hyperspecial point. Let \( \mathcal{C} \) denote the set of \( G \)-conjugacy classes of

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pairs \((H, x)\) where \(H\) is a maximal rank unramified subgroup in \(G\) and \(x\) is a hyperspecial point in \(B^\text{red}(H)\).

Let \(I\) denote the set of pairs \((F, H)\) where \(F\) is a facet in \(B(G)\) and \(H\) is a maximal rank connected reductive \(\mathfrak{f}\)-subgroup in \(G_F\). In \(\S 3.2\) we define on \(I\) an equivalence relation, denoted \(\sim\).

In \(\S 3.3\) we associate to each element \((F, H) \in I\) a conjugacy class \(C(F, H)\) of pairs \((H, x)\) where \(H\) is a maximal rank unramified subgroup of \(G\) and \(x\) is a hyperspecial point in \(B^\text{red}(H)\).

The set \(I\) is too large, so we restrict our attention to the subset \(I^c\) of \(f\)-cuspidal pairs in \(I\). A pair \((F, H) \in I\) is said to be \(f\)-cuspidal if the maximal \(f\)-split torus in \(H\) coincides with the maximal \(f\)-split torus in the center of \(G_F\). (Equivalently, \(H\) is \(f\)-cuspidal in \(G_F\) if and only if \(H\) lies in no proper \(f\)-parabolic subgroup of \(G_F\).)

We now state Theorem 3.4.1, the main result of this paper.

**Theorem.** There is a bijective correspondence between \(I^c / \sim\) and \(C\) given by the map which sends \((F, H)\) to \(C(F, H)\).

A torus in \(G\) is called unramified if it is an unramified subgroup of \(G\). From Remark 2.1.2, every maximal unramified torus of \(G\) is a full rank unramified subgroup of \(G\). Let \(I^m\) denote the set of pairs \((F, T) \in I^c\) such that \(T\) is a maximal \(f\)-torus in \(G_F\). Let \(C^T\) denote the set of conjugacy classes of maximal unramified tori in \(G\). Since the reduced Bruhat-Tits building of a torus is a point, we immediately derive the following corollary.

**Corollary.** There is a bijective correspondence between \(I^m / \sim\) and \(C^T\) given by the map which sends \((F, T)\) to \(C(F, T)\).

If our group is connected, reductive, and \(k\)-split, then this Corollary can be derived from some work of Paul Gérardin [8]. If our group is connected, reductive, and unramified, then Waldspurger [13] stated a variant of this Corollary as a hypothesis.

We remark that if \(\mathfrak{f}\) is algebraically closed, then \(C\) and \(I^m / \sim\) both have one element. In this case, the element of \(C\) is the conjugacy class of maximal \(k\)-split tori in \(G\), and \(I^m\) consists of those pairs \((F, T)\) where \(F\) is an alcove in \(B(G)\) and \(T\) is a maximal torus in \(G_F\).

Let \(K\) denote a fixed maximal unramified extension of \(k\). From Remark 2.1.2 a maximal unramified torus in \(G\) is the group of \(k\)-rational points of a maximal \(K\)-split torus in \(G\) which is defined over \(k\). From a theorem of Steinberg, \(G^\circ\) is quasisplit over \(K\). Thus, the centralizer in \(G^\circ\) of a maximal unramified torus in \(G\) is the group of \(k\)-rational points of a maximal \(k\)-torus in \(G\). Since this correspondence is one-to-one, our theorem also provides a classification of the \(G\)-conjugacy classes of maximal \(k\)-tori of \(G\) which arise in this way.

Finally, in the case when \(\mathfrak{f}\) is finite and \(G\) is unramified, we give a description of the stable conjugacy classes of maximal unramified tori in \(G\). I thank Bob Kottwitz for telling me that such a description ought to be possible.

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1. NOTATION

In addition to the notation discussed in the introduction, we will require the following.

1.1. Basic notation. Let \( k \) denote a field with nontrivial discrete valuation \( \nu \). We assume that \( k \) is complete and the residue field \( \mathfrak{f} \) is perfect.

Let \( K \) be a fixed maximal unramified extension of \( k \). Let \( \mathfrak{F} \) denote the residue field of \( K \). Note that \( \mathfrak{F} \) is an algebraic closure of \( \mathfrak{f} \).

Let \( G \) be a linear algebraic group defined over \( k \). We assume that the identity component \( G^\circ \) of \( G \) is reductive. We let \( G = G(k) \), the group of \( k \)-rational points of \( G \). Let \( G^\circ = G^\circ(k) \). \( DG^\circ \) will denote the group of \( k \)-rational points of the derived group \( DG^\circ \) of \( G^\circ \).

When we talk about a torus in \( G \), we mean the group of \( k \)-rational points of a \( k \)-torus in \( G^\circ \).

In order to avoid a proliferation of superscripts, we adopt the following convention. We shall call a subgroup of \( G \) a parabolic subgroup of \( G \) provided that it is a parabolic subgroup of \( G^\circ \). We adopt a similar convention with respect to tori and Levi subgroups.

If \( g, h \in G \), then \( ghg^{-1} \).

If a group \( L \) acts on a set \( S \), then \( S^L \) denotes the set of \( L \)-fixed points of \( S \).

1.2. Apartments, buildings, and associated notation. Let \( B(G) = B(G, k) \) denote the (enlarged) Bruhat-Tits building of \( G^\circ \). We identify \( B(G) \) with the \( \Gamma \)-fixed points of \( B(G, K) \), the Bruhat-Tits building of \( G^\circ(K) \). Let \( B^{\text{red}}(G) = B(DG^\circ, k) \) denote the reduced Bruhat-Tits building of \( G^\circ \). According to [11], we have a decomposition \( B(G, k) = B^{\text{red}}(G, k) \times B(Z(G), k) \) where \( Z(G) \) denotes the center of \( G \).

For a \( k \)-Levi subgroup \( M \) of \( G \), we identify \( B(M, k) \) in \( B(G, k) \). There is not a canonical way to do this, but every natural embedding of \( B(M, k) \) in \( B(G, k) \) has the same image.

Given a maximal \( k \)-split torus \( S \) of \( G \) which is defined over \( k \) we have the torus \( S = S(k) \) in \( G \) and the corresponding apartment \( A(S) = A(S, k) \) in \( B(G) \). Let \( T \) be a maximal \( K \)-split \( k \)-torus of \( G \) containing \( S \) [3, Corollaire 5.1.12]. We identify \( A(S, k) \) with \( A(T, K)^T \).

For \( \Omega \subset A(S) \), we let \( A(A(S), \Omega) \) denote the smallest affine subspace of \( A(S) \) containing \( \Omega \).

Suppose \( x \in B(G) \). We will denote the parahoric subgroup of \( G^\circ \) attached to \( x \) by \( G_x \), and we denote its pro-unipotent radical by \( G^+_x \). Note that both \( G_x \) and \( G^+_x \) depend only on the facet of \( B(G) \) to which \( x \) belongs. If \( F \) is a facet in \( B(G) \) and \( x \in F \), then we define \( G_F = G_x \) and \( G^+_F = G^+_x \). Recall that \( G_x \) is a subgroup of \( \text{stab}_{G^\circ}(x) \). For a facet \( F \) in \( B(G) \) the quotient \( G_F/G^+_F \) is the group of \( \mathfrak{f} \)-rational points of a connected reductive group \( G_F \) defined over \( \mathfrak{f} \).

We denote the parahoric subgroup of \( G^\circ(K) \) corresponding to \( x \in B(G, K) \) by \( G(K)_x \). We denote the pro-unipotent radical of \( G(K)_x \) by \( G(K)^+_x \). The subgroups \( G(K)_x \) and \( G(K)^+_x \) depend only on the facet of \( B(G, K) \) to which \( x \) belongs. If \( F \) is a facet in \( B(G, K) \) and \( x \in F \), then we define \( G(K)_F = G(K)_x \) and \( G(K)^+_F = G(K)^+_x \). For a facet \( F \) in \( B(G, K) \), the quotient \( G(K)_F/G(K)^+_F \) is the group of \( \mathfrak{F} \)-rational points of a connected, reductive \( \mathfrak{F} \)-group \( G_F \).
Suppose $F$ is a $\Gamma$-invariant facet in $\mathcal{B}(G, K)$. In this case, $F' = F^\Gamma$ is a facet in $\mathcal{B}(G)$. Moreover, we have $G_{F'} = (G(K)_F)^\Gamma$, $G_{F'}^+ = (G(K)_F^+)^\Gamma$, and $G_F = G_{F'}$ (in particular, $G_F$ is defined over $f$). Sometimes, we will abuse notation and denote by $G_F$ (resp., $G_{F'}^+$, resp., $G(K)_F^+$, resp., $G(K)_F^+$) the group $G_{F'}$ (resp., $G_{F'}^+$, resp., $G(K)_F^+$, resp., $G(K)_F^+$).

A vertex $x \in \mathcal{B}^{\text{red}}(G)$ is said to be hyperspecial provided that $x$ is a $\Gamma$-invariant special vertex in $\mathcal{B}^{\text{red}}(G, k)$. Equivalently, a vertex $x \in \mathcal{B}^{\text{red}}(H, k)$ is called hyperspecial provided that the $\mathfrak{g}$-root system of $G_x$ and the root system of $G$ are isomorphic; in particular, this implies that $G$ is $K$-split.

If there exists a hyperspecial vertex in $\mathcal{B}^{\text{red}}(G)$, then the group $G$ is called unramified.

2. Maximal rank subgroups over $k$ and $f$

In this section we show how to move between maximal rank unramified subgroups over $k$ and maximal rank connected reductive subgroups over $f$.

2.1. Maximal unramified subgroups. We recall that a subgroup $H$ of $G$ is unramified if $H$ is the group of $k$-rational points of a connected reductive $k$-subgroup $H$ of $G$ and $\mathcal{B}^{\text{red}}(H, k)$ contains a hyperspecial point. We shall also call such an $H$ an unramified subgroup of $G$. The following result will be used throughout the remainder of the paper.

**Lemma 2.1.1.** Suppose $H$ is a maximal rank connected reductive $K$-split $k$-subgroup of $G$. Every maximal $K$-split $k$-torus in $H$ is a maximal $K$-split torus in $G$.

**Proof.** Fix a maximal $K$-split $k$-torus $T$ in $H$. (The existence of such a torus follows from [3, Corollaire 5.1.12].)

Let $M = C_G \cdot (T)$. Then $M$ is a $K$-Levi subgroup of $G$ which is defined over $k$. Let $S'$ be a maximal $k$-split torus in $M$. From [3, Corollaire 5.1.12], there exists a maximal $K$-split torus $S \subset M$ such that $S' \subset S$ and $S$ is defined over $k$. Note that $S$ is also a maximal $K$-split torus in $G$. Since $S \subset M$, we have $T \subset S$. It is enough to show that $S = T$. Note that $H \cdot S$ is a connected, reductive $K$-split $k$-subgroup of $G$ and the rank of $H \cdot S$ equals the rank of $S$. However, the rank of $S$ is greater than or equal to that of $T$. Since $H$ was chosen to have maximal rank, we conclude that $S = T$. \hfill \Box

**Remark 2.1.2.** In particular, the above lemma shows that every maximal $K$-split $k$-torus in $G$ is a maximal $K$-split torus in $G$.

2.2. Some results about maximal unramified tori. We collect some facts concerning maximal unramified tori.

**Lemma 2.2.1.** Suppose $T$ is a maximal $K$-split torus in $G$ which is defined over $k$. Let $T$ denote the group of $k$-rational points of $T$. There is a maximal $k$-split torus $S$ in $G$ such that $\mathcal{B}(T)$ is an affine subspace of $A(S, k)$.

**Proof.** From [1, Proposition 8.15] we can write $T = T_s \cdot T_a$ where $T_s$ the maximal $k$-split torus in $T$ and $T_a$ is the maximal $k$-anisotropic subtorus of $T$. Let $M = C_G \cdot (T_s)$. Then $T \subset M$.

\footnote{Check in Corvalis to make sure that you’ve got this 100%}
and $M$ is a $k$-Levi subgroup. Let $M$ denote the group of $k$-rational points of $M$. We have that the image of $B(T)$ in $B^\text{red}(M)$ is a point, call it $x_T$. Let $S$ be a maximal $k$-split torus in $M$ such that the image (apartment) of $A(S, k)$ in $B^\text{red}(M)$ contains $x_T$. Since $T_s \subset S$, we have $B(T) \subset A(S, k)$. □

**Lemma 2.2.2.** Suppose $T_1$ and $T_2$ are maximal $K$-split tori of $G$ which are defined over $k$. If $F$ is a $\Gamma$-invariant facet in $A(T_1, K) \cap A(T_2, K)$ and the images of $T_1(K) \cap G(K)_F$ and $T_2(K) \cap G(K)_F$ in $G_F(\mathfrak{F})$ coincide, then $T_1$ and $T_2$ are $G^+_F$-conjugate.

**Proof.** Let $T$ denote the maximal $\mathfrak{f}$-torus in $G_F$ whose group of $\mathfrak{F}$-rational points is the image of $T_1(K) \cap G(K)_F$ in $G_F(\mathfrak{F})$. Note that $T$ is defined over $\mathfrak{f}$.

Let $Z$ denote the centralizer of $T_1$ in $G^\circ$. The group $Z$ is a $K$-Levi subgroup (and maximal $k$-torus) of $G$ which is defined over $k$. Note that $B(Z, K) = A(T_1, K)$ and so for all facets $F$ in $A(T_1, K)$, we have $Z(K)_F = Z(K) \cap G(K)_F$ and $Z(K)_F^{+} = Z(K) \cap G(K)_F^{+}$.

There exists an $h \in G(K)_F$ such that $hT_1 = T_2$. Let $\bar{h}$ denote the image of $h$ in $G_F(\mathfrak{F})$. By hypothesis, $\bar{h}T = T$. Thus, $\bar{h} \in (N_G(T))(\mathfrak{F})$. Consequently, there exist $n \in (N_G(T_1))(K) \cap G(K)_F$ and $g \in G(K)_F^{+}$ such that $h = g n$. We have $T_2 = hT_1 = gT_1$.

For $\gamma \in \Gamma$, let $c_g(\gamma) := g^{-1}(g)g$; $c_g$ is a one-cocycle. We will show that $c_g(\gamma) \in Z(K)_F^{+}$ for all $\gamma \in \Gamma$. Fix $\gamma \in \Gamma$. Since $F$ is $\Gamma$-stable and $g \in G(K)_F^{+}$, we have $c_g(\gamma) \in G(K)_F^{+}$.

Since $c_g(\gamma)T_1 = T_1$, we have $c_g(\gamma) \in N_{G^0}(T_1)(K)$. Thus $A(T_1, K)$ is $c_g(\gamma)$-stable. If $C$ is an alcove in $A(T_1, K)$ such that $F \subset C$, then $c_g(\gamma)$ fixes $C$ point-wise and therefore $c_g(\gamma)$ fixes $A(T_1, K)$. Thus, we conclude that $c_g(\gamma) \in Z(K)_F^{+}$.

Since $H^1(\Gamma, Z(K)_F^{+})$ is trivial, there exists $z \in Z(K)_F^{+}$ such that $gz$ is fixed by $\Gamma$. We have $g^zT_1 = T_2$ and $gz \in (G(K)_F^{+})^\Gamma = G_F^\Gamma$. □

Suppose $(F, T) \in I$ with $T$ a torus. Let $F'$ be the facet in $B(G, K)$ whose set of $\Gamma$-fixed points is $F$. In the final paragraph of the proof of [3, Proposition 5.1.10] Bruhat and Tits use [7, Exp. XI, Cor. 4.2] to show that there exists a maximal $K$-split torus $T$ in $G$ such that $T$ is defined over $k$, the apartment $A(T, K)$ contains $F$, and the image of $T(K) \cap G(K)_F$ in $G_F(\mathfrak{F}) = G_F^\Gamma(\mathfrak{F})$ is $T(\mathfrak{F})$. We record this result in the following lemma.

**Lemma 2.2.3.** If $(F, T) \in I$ with $T$ a torus, then there exists a maximal $K$-split torus $T$ in $G$ such that $T$ is defined over $k$, the apartment $A(T, K)$ contains $F$, and the image of $T(K) \cap G(K)_F$ in $G_F(\mathfrak{F})$ is $T(\mathfrak{F})$. □

2.3. **From maximal rank unramified subgroups of $G$ to connected reductive $\mathfrak{f}$-groups.** Suppose that $H$ is a maximal rank unramified subgroup of $G$. We identify $B(H, K)$ with its image in $B(G, K)$. (As usual, there does not exist a canonical embedding of $B(H, K)$ in $B(G, K)$, but the image of any natural embedding is independent of the embedding.) We therefore have

$$B(H) = B(H, K)^\Gamma \subset B(G, K)^\Gamma = B(G).$$

We now collect some facts about $B(H)$.

**Lemma 2.3.1.** Suppose $H$ is a maximal rank unramified subgroup of $G$. Let $H$ denote the group of $k$-rational points of $H$. 
Remark Z is contained in the center of G. Since Z and let H be a subgroup in B which is contained in B from the work of Bruhat and Tits [2, 3].

Proof. “(1)”: Since B(H) is the Bruhat-Tits building of H, the first half of the statement follows from the work of Bruhat and Tits [2, 3].

For any Γ-invariant facet F of B(G, K), we have FΓ = F ∩ B(G) is a facet of B(G). Consequently, for any Γ-invariant facet F of B(H, K) ⊂ B(G, K), we have that FΓ is a facet of B(G) which is contained in B(H).

“(2)”: Suppose F is a facet in B(H). Let H be the maximal rank connected reductive f-subgroup in GF corresponding to the image of H(K) ∩ G(K)F in GF(F). We have (F, H) ∈ I.

Now suppose that F is a maximal facet in the preimage in B(H) of a facet in Bred(H). Choose a subgroup H in GF as in the previous paragraph.

Let S′ be a maximal k-split torus in H so that F ⊂ A(S′, k) ⊂ B(H, k). Let S be a maximal k-split torus in B(G) such that S′ ⊂ S. Let Z denote the maximal k-split torus in the center of H and let Z ⊂ GF denote the split f-torus whose group of F-rational points coincides with the image of Z(K) ∩ G(K)F in GF(F). We have

Z ⊂ S′ ⊂ S

and we may, in the natural way, identify

B(Z, k) ⊂ B(S′, k) ⊂ B(S, k) = A(S, k).

Since F is a maximal facet in the preimage in B(H) = Bred(H) × B(Z, k), for all affine roots ψ of G with respect to S, k, and ν, if ψ is constant on F, then ψ is constant on B(Z, k). Therefore, Z is contained in the center of GF and so H cannot lie in a proper parabolic f-subgroup of GF.

“(3)”: This is clear. □

Remark 2.3.2. We maintain the notation used in the proof of (2) above.

(1) The torus Z is the maximal f-split torus in the center of GF exactly when F is a maximal facet in the preimage in B(H) of a vertex in Bred(H).

(2) If H is a maximal K-split k-torus in G, then H is a maximal f-torus in GF.

The previous lemma gives us a way to associate to a pair (H, x), with H a maximal rank unramified subgroup in G and x ∈ Bred(H) hyperspecial, an element of Ic.

2.4. From subgroups over f to unramified subgroups over k.

Lemma 2.4.1. Suppose H1 and H2 are maximal rank unramified subgroups of G. If F is a Γ-invariant facet in B(H1, K) ∩ B(H2, K) which projects to a Γ-fixed special vertex in Bred(H1, K)
Proof. Let $H$ denote the maximal rank reductive subgroup in $G_F$ whose group of $\mathfrak{F}$-rational points is the image of $H_1(K) \cap G(K)_F$ in $G_F(\mathfrak{F})$. Note that $H$ is defined over $\mathfrak{F}$. Let $T$ be a maximal $\mathfrak{F}$-torus in $H$ which contains a maximal $\mathfrak{F}$-split torus of $H$. Let $\Phi(H, T)$ denote the $\mathfrak{F}$-root system of $H$ with respect to $T$. As in Lemma 2.2.3, we choose a maximal $K$-split $k$-torus $T_i$ in $H_i$ lifting $T$. Note that, since the image of $F$ in $B^{\text{red}}(H_i, k)$ is hyperspecial, $H_i$ is completely determined by $\Phi(H, T)$ and $T_i$. From Lemma 2.2.2, we conclude that $H_1$ and $H_2$ are $G_F^+$-conjugate. \qed

Lemma 2.4.2. Suppose $F$ is a facet in $B(G)$ and $H$ is a maximal rank connected reductive $\mathfrak{F}$-subgroup of $G_F$. There exists a maximal rank unramified $H$ in $G$ such that:

1. The facet $F$ belongs to $B(H, k)$.

2. The image of $H(K) \cap G(K)_F$ in $G_F(\mathfrak{F})$ is the group of $\mathfrak{F}$-rational points of $H$.

3. The image of $F$ in $B^{\text{red}}(H, k)$ is a special vertex, $x_F$.

Proof. Let $T$ be a maximal $\mathfrak{F}$-torus in $H$ (and hence in $G_F$) which contains a maximal $\mathfrak{F}$-split torus of $H$. Let $\Phi(H, T)$ denote the $\mathfrak{F}$-root system of $H$ with respect to $T$. As in Lemma 2.2.3, let $T$ be a lift of $T$ to a maximal $K$-split $k$-torus $T$ in $G$. We think of $\Phi(H, T)$ as a subset of $\Phi(G, T)$, the $K$-root system of $G$ with respect to $T$. Let $H$ be the $K$-split full rank subgroup of $G$ whose group of $K$-rational points is generated by $T(K)$ and the root groups in $G(K)$ corresponding to elements of $\Phi(H, T)$. Note that $H$ is defined over $k$, and, by construction, the image of $F \subset B(H, K)$ in $B^{\text{red}}(H, K)$ is a $\Gamma$-fixed special vertex. The lemma follows. \qed

Remark 2.4.3. If, in the statement of Lemma 2.4.2, $H$ is also assumed to be $\mathfrak{F}$-cuspidal, then $F$ is a maximal facet in the preimage in $B(H, k)$ of $x_F$.

3. The parameterization

In this section, we present a parameterization of $C$ via Bruhat-Tits theory.

3.1. Strong associativity. Following [9, 10], in [5, §2.3] the concept of strong associativity is developed. We recall the definition and some of its consequences.

Definition 3.1.1. Two facets $F_1$ and $F_2$ of $B(G)$ are strongly associated if for all apartments $\mathcal{A}$ containing $F_1$ and $F_2$, we have

$$A(\mathcal{A}, F_1) = A(\mathcal{A}, F_2).$$

Remark 3.1.2. Two facets $F_1, F_2$ of $B(G)$ are strongly associated if and only if there exists an apartment $\mathcal{A}$ containing $F_1$ and $F_2$ such that $A(\mathcal{A}, F_1) = A(\mathcal{A}, F_2)$. See [5, Lemma 2.3.3].

Remark 3.1.3. Suppose $F_1$ and $F_2$ are strongly associated facets in $B(G)$. There is an identification of $G_{F_1}$ with $G_{F_2}$. Namely, the natural $\Gamma$-equivariant map

$$G(K)_{F_1} \cap G(K)_{F_2} \rightarrow G_{F_1}(\mathfrak{F})$$

is surjective with kernel $G(K)_{F_1} \cap G(K)_{F_2} = G(K)_{F_1} \cap G(K)_{F_2} = G(K)_{F_1} \cap G(K)_{F_2}$. See, for example, [5, Lemma 2.5.1].
Definition 3.1.4. If $F_1$ and $F_2$ are strongly associated facets in $B(G)$, then we denote the natural identification of $G_{F_1}$ and $G_{F_2}$ introduced above by $G_{F_1} \overset{\sim}{=} G_{F_2}$.

3.2. An equivalence relation on $I$. We first consider the action of $G$ on $I$. Suppose $g \in G$ and $(F, H) \in I$. From Lemma 2.4.2 there exists a maximal rank unramified subgroup $H$ of $G$ such that the building of $B(H)$ contains $F$, the image of $H(K) \cap G(K)_F$ in $G_F(\mathfrak{F})$ is $H(\mathfrak{F})$, and the image of $F$ in $B^\text{red}(H)$ is a hyperspecial vertex. Define

$$g(F, H) := (gF, g^H)$$

where $g^H$ is the maximal rank connected reductive $F$-group in $G_{gF}$ whose group of $\mathfrak{F}$-rational points coincides with the image of $g^H(K) \cap G(K)_{gF}$ in $G_{gF}(\mathfrak{F})$. From Lemma 2.4.1, this definition is independent of the unramified subgroup $H$ we choose to represent $H$.

We are now prepared to introduce a relation on $I$.

Definition 3.2.1. Suppose $(F_1, H_1)$ and $(F_2, H_2)$ are two elements of $I$. We will write $(F_1, H_1) \sim (F_2, H_2)$ provided that there exist an apartment $A$ in $B(G)$ and $g \in G$ such that

1. $\emptyset \neq A(A_1, F_1) = A(A_2, gF_2)$ and
2. $H_1 \overset{i}{=} g^H_2$ in $G_{F_1} \overset{i}{=} G_{gF_2}$.

Lemma 3.2.2. The relation $\sim$ on $I$ is an equivalence relation.

Proof. We will verify that the relation is transitive. The proofs that the relation is reflexive and symmetric are easier and left to the reader.

Suppose $(F_i, H_i) \in I$ for $i = 1, 2, 3$. Suppose $(F_1, H_1) \sim (F_2, H_2)$ and $(F_2, H_2) \sim (F_3, H_3)$. We want to show $(F_1, H_1) \sim (F_3, H_3)$.

There exist $g_2, g_3 \in G$ and apartments $A_{12}$ and $A_{23}$ in $B(G)$ such that

1. $\emptyset \neq A(A_{12}, F_1) = A(A_{12}, g_2F_2)$
2. $\emptyset \neq A(A_{23}, F_2) = A(A_{23}, g_3F_3)$

and

1. $H_1 \overset{i}{=} g^H_2$ in $G_{F_1} \overset{i}{=} G_{g_2F_2}$
2. $H_2 \overset{i}{=} g^H_3$ in $G_{F_2} \overset{i}{=} G_{g_3F_3}$

Since $g_2F_2 \subset A_{12} \cap g_2A_{23}$, there exists an element $h \in G_{g_2F_2}$ such that $hg_2A_{23} = A_{12}$. We have

$$\emptyset \neq A(A_{12}, F_1) = A(A_{12}, g_2F_2) = A(hg_2A_{23}, h g_2 F_2) = h g_2 A(A_{23}, F_2) = h g_2 A(A_{23}, g_3 F_3) = A(A_{12}, h g_2 g_3 F_3).$$

Moreover, we have that $G_{F_1} \cap G_{g_2F_2} \cap G_{h g_2 g_3 F_3}$ surjects, under the natural map, onto $G_{F_1}(f)$ (resp., $G_{g_2F_2}(f)$, resp., $G_{h g_2 g_3 F_3}(f)$). Thus, there exists $h' \in G_{F_1} \cap G_{g_2F_2} \cap G_{h g_2 g_3 F_3}$ such that

$$H \overset{i}{=} g^H_2 \overset{i}{=} h' g^H_2 \overset{i}{=} h' h g^H_3 \text{ in } G_{F_1} \overset{i}{=} G_{g_2F_2} \overset{i}{=} G_{g_2F_2} \overset{i}{=} G_{h g_2 g_3 F_3}. \quad \Box$$
3.3. **A map from $I/ \sim$ to $C$.** From Lemmas 2.4.1 and 2.4.2, the following definition makes sense.

**Definition 3.3.1.** Suppose $(F, H) \in I$. Let $H$ be any maximal unramified subgroup of $G$ such that the building $B(H, K)$ contains $F$, the image of $H(K) \cap G(K)_F$ in $G_F(\mathfrak{F})$ is $H(\mathfrak{F})$, and the image of $F$ in $B^{\text{red}}(H, k)$ is a hyperspecial vertex $x_F$. Define $C(F, H) \in C$ by setting $C(F, H)$ equal to the $G$-conjugacy class of the pair $(H(k), x_F)$.

**Remark 3.3.2.** If $g \in G$ and $(F, H) \in I$, then $C(F, H) = C(gF, gH)$.

**Lemma 3.3.3.** The map from $I$ to $C$ which sends $(F, H) \in I$ to $C(F, H)$ induces a well-defined map from $I/ \sim$ to $C$.

**Proof.** Suppose $(F_1, H_1)$ and $(F_2, H_2)$ are two elements of $I$. We need to show that if $(F_1, H_1) \sim (F_2, H_2)$, then $C(F_1, H_1) = C(F_2, H_2)$.

Since $(F_1, H_1) \sim (F_2, H_2)$, there exist $g \in G$ and an apartment $\mathcal{A}$ in $B(G)$ such that

$$\emptyset \neq A(\mathcal{A}, F_1) = A(\mathcal{A}, gF_2)$$

and

$$H_1 \overset{i}{=} gH_2 \text{ in } G_{F_1} \overset{i}{=} G_{F_2}.$$ 

From Remark 3.3.2, we can assume that $g = 1$.

From Lemma 2.4.2 there exists a maximal rank unramified subgroup $H_2$ of $G$ such that $F_2 \subset B(H_2, K)$, the image of $H_2(K) \cap G(K)_{F_2}$ in $G_{F_2}(\mathfrak{F})$ coincides with $H_2(\mathfrak{F})$, and the image of $F_2$ in $B^{\text{red}}(H, k)$ is hyperspecial. Note that $C(F_2, H_2)$ is the $G$-conjugacy class of $(H_2(k), x_{F_2})$ where $x_{F_2}$ is the image of $F_2$ in $B^{\text{red}}(H)$. Let $T_2$ be a maximal $\mathfrak{f}$-torus in $H_2$. From Lemma 2.2.3 we can choose a maximal $K$-split $k$-torus $T_2$ in $H_2$ lifting $T_2$ such that $F_2 \subset B(T_2, k)$. It follows from Lemma 2.2.1 that we can choose $h \in G_{F_2}$ such that $B(hT_2, k) \subset \mathcal{A}$. Since $\emptyset \neq A(\mathcal{A}, F_1) = A(\mathcal{A}, F_2) \subset B(hT_2, k)$, we conclude that $F_1 \subset B(hT_2, k) \subset B(hH_2, k)$.

Let $H'$ denote the maximal rank connected reductive $\mathfrak{f}$-subgroup in $G_{F_1}$ such that the image of $hH_2(K) \cap G(K)_{F_1}$ in $G_{F_1}(\mathfrak{F})$ coincides with $H'(\mathfrak{F})$. We have

$$H' \overset{i}{=} hH_2 \text{ in } G_{F_1} \overset{i}{=} G_{F_2}$$

and

$$H_1 \overset{i}{=} H_2 \text{ in } G_{F_1} \overset{i}{=} G_{F_2}.$$ 

Thus, there exists $h' \in G_{F_1} \cap G_{F_2}$ such that

$$h' H_1 \overset{i}{=} h' H_2 \overset{i}{=} h H_2 \overset{i}{=} H' \text{ in } G_{F_1} \overset{i}{=} G_{F_2} \overset{i}{=} G_{F_2} \overset{i}{=} G_{F_1}.$$ 

In other words, $h' H_1 = H'$ in $G_{F_1}$. We conclude from Lemma 2.4.1 that $C(F_1, H_1)$ is the $G$-conjugacy class of $(h')^{-1}hH_2(k)$, i.e., $C(F_1, H_1) = C(F_2, H_2)$. 

$\square$
3.4. A bijective correspondence. We now prove the main result of this paper.

**Theorem 3.4.1.** There is a bijective correspondence between $I^c / \sim$ and $\mathcal{C}$ given by the map sending $(F, H)$ to $C(F, H)$.

*Proof.* From Lemma 3.3.3, this map is well defined. From Lemma 2.3.1 (2) and Lemma 2.4.2 the map is surjective. It remains to show that the map is injective.

Suppose $(F_1, H_1)$ and $(F_2, H_2)$ are pairs in $I^c$ such that $C(F_1, H_1) = C(F_2, H_2)$. We need to show that $(F_1, H_1) \sim (F_2, H_2)$.

For $i = 1, 2$, from Lemma 2.4.2 we can choose a maximal unramified subgroup $H_i$ in $G$ such that the building $B(H_i, K)$ contains $F_i$, the image of $H_i(K) \cap G(K)_F$ in $G_{F_i}(\mathfrak{f})$ is $H_i(\mathfrak{f})$, the image of $F_i$ in $B_{\text{red}}(H)$ is hyperspecial, and the $G$-conjugacy class of the pair $(H_i(k), x_{F_i})$ is $C(F_i, H_i)$. Since $C(F_1, H_1) = C(F_2, H_2)$, there exists a $g \in G$ such that $g \cdot H_2 = H_1$ and $g \cdot x_{F_2} = x_{F_1}$ in $B_{\text{red}}(H_1, k)$. Let $H = g \cdot H_2 = H_1$ and let $H = H(k)$.

Note that both $F_1$ and $g \cdot F_2$ lie in $B(H)$. Moreover, both $F_1$ and $g \cdot F_2$ lie in the preimage in $B(H)$ of $x_{F_1} \in B_{\text{red}}(H)$. Since $(F_1, H_1)$ is an $\mathfrak{f}$-cuspidal pair, from Remark 2.4.3 we have that $F_1$ is a maximal facet in the preimage in $B(H)$ of $x_{F_1} \in B_{\text{red}}(H)$. Similarly, $g \cdot F_2$ is a maximal facet in the preimage in $B(H)$ of $x_{F_1} \in B_{\text{red}}(H)$. From Lemma 2.3.1 (3), the facets $F_1$ and $g \cdot F_2$ are strongly associated. Since the image of $H(K) \cap G(K)_{F_1} \cap G(K)_{g \cdot F_2}$ in $G_{F_1}(\mathfrak{f})$ (resp., $G_{g \cdot F_2}(\mathfrak{f})$) is $H_1(\mathfrak{f})$ (resp., $g \cdot H_2(\mathfrak{f})$), we have

$$H_1 \sim g \cdot H_2 \text{ in } G_{F_1} \sim g \cdot G_{F_2}.$$

We immediately have:

**Corollary 3.4.2.** There is a bijective correspondence between $I^m / \sim$ and $\mathcal{C}^T$ given by the map sending $(F, T)$ to $C(F, T)$.

4. Stable conjugacy classes of maximal unramified tori

We say that two maximal $k$-tori $T_1$ and $T_2$ in $G$ are *stably conjugate* provided that there exists $g \in G(\tilde{k})$ such that $T_i(k) = g(T_2(k))$. This is equivalent to saying that there exists a strongly regular element $t \in T_2(k)$ (that is, a regular semisimple element whose centralizer is connected) such that $g \cdot t \in T_1(k)$.

For the remainder of this paper, we assume that $\mathfrak{f}$ is finite and $G$ is unramified. From for example [12, ??], this is equivalent to assuming that $\mathfrak{f}$ is finite and $G$ is $k$-quasisplit and $K$-split. Under these assumptions, we have that every maximal unramified torus of $G$ is also an unramified maximal torus of $G$, and *vice-versa*.

Suppose that $T_i$ ($i = 1, 2$) is a maximal unramified torus in $G$. From Hilbert’s Theorem 90, if $t_1 \in T_1(k)$ and $t_2 \in T_2(k)$ are strongly regular elements which are conjugate by an element of $G(\tilde{k})$, then $t_1$ and $t_2$ are conjugate by an element of $T(K)$. Consequently, two maximal unramified tori $T_1$ and $T_2$ of $G$ are stably conjugate if and only if there exists a $g \in G(K)$ such that $T_1(k) = g(T_2(k))$.

The goal of this section is to provide a nice parameterization of the stable conjugacy classes of maximal unramified tori in $G$. 
Thus, it is enough to show that

Fix a strongly regular element \( \sigma \) such that

we conclude that \( g \) and \( g' \) are maximal unramified tori in \( G \) and \( G' \) respectively. Let \( W \) denote the Weyl group \( N_G(T)(K)/T(K) \) which is naturally isomorphic to \( N_{G'}(T)({\mathfrak{g}})/T({\mathfrak{g}}) \).

Choose a topological generator \( \sigma \in \Gamma \) for \( \Gamma \). Since \( \sigma \) preserves \( T \), it acts on \( W \). Two elements \( w_1, w_2 \in W \) are \( \sigma \)-conjugate provided that there exists \( w' \in W \) such that \( w'w_1\sigma(w')^{-1} = w_2 \). One can check that \( \sigma \)-conjugacy defines an equivalence relation on \( W \). The partitions associated to this equivalence relation are called \( \sigma \)-conjugacy classes.

**Theorem 4.0.3.** Suppose \( \mathfrak{f} \) is finite and \( G \) is unramified. The stable conjugacy classes of maximal unramified tori in \( G \) are in natural bijective correspondence with the \( \sigma \)-conjugacy classes in \( W \).

**Proof.** Since there is a single \( G(K) \)-conjugacy class of maximal \( K \)-split tori in \( G \), all maximal unramified tori in \( G \) are \( G(K) \)-conjugate to \( T \). If \( g \in G(K) \) and \( gT \) is a maximal unramified torus in \( G \), then \( gT = \sigma(gT) = \sigma(g)T \). Consequently, \( \sigma^{-1}g \in N_G(T)(K) \). Note that if \( g, g' \in G(K) \) such that \( gT = g'T \) and \( gT \) is defined over \( k \), then there exists an \( n \in N_G(T)(K) \) such that \( g = g'n \) and so \( \sigma^{-1}g = \sigma(n)^{-1}\sigma(g')^{-1}g'n \). In this way, we get a well-defined map \( \omega \) from maximal unramified tori in \( G \) to \( \sigma \)-conjugacy classes in \( W \); namely, \( \omega(gT) \) is the \( \sigma \)-conjugacy class of \( \sigma^{-1}gT(K) \).

Suppose \( g, g' \in G(K) \). We first show that if \( gT \) and \( g'T \) are two stably conjugate maximal unramified tori in \( G \), then \( \omega(gT) = \omega(g'T) \). Since \( gT \) and \( g'T \) are stably conjugate, there exist \( h \in G(K) \) and a strongly regular \( t \in T(K) \) such that \( gT = (gT)(k) \) and \( hT = (g'T)(k) \). This implies that \( \sigma(h)^{-1}h \in (gT)(K) \). Since

\[
\sigma(hg)^{-1}h \in (gT)(K) = (\sigma(g)^{-1}h) \cdot gT(K)
\]

we conclude that \( \omega(gT) = \omega(hgT) \). Since \( hgT = g'T \), from the previous paragraph we have \( \omega(hgT) = \omega(g'T) \). Consequently, \( \omega(gT) = \omega(g'T) \).

Therefore, we have a map from stable conjugacy classes of maximal unramified tori of \( G \) to \( \sigma \)-conjugacy classes in \( W \). We now show that this map is injective. Suppose \( g, g' \in G(K) \) such that \( gT \) and \( g'T \) are maximal unramified tori in \( G \) and \( \omega(gT) = \omega(g'T) \). By replacing \( g' \) with \( g'n \) for some \( n \in N_G(T)(K) \), we may assume that

\[
\sigma(g)^{-1}g = \sigma(g')^{-1}g'T(K)
\]

Fix a strongly regular element \( t \in (gT)(k) \). It will be enough to show that \( g'g^{-1}t \in (g'T)(k) \). Thus, it is enough to show

\[
g'g^{-1}t = \sigma(g')\sigma(g)^{-1}t
\]

However, Equation (2) is valid if and only if

\[
\sigma(g)^{-1}t = (\sigma(g')^{-1}g')^{-1}t.
\]
Since \( g^{-1}t \in T(K) \), it follows from Equation (1) that the map is injective.

Finally, we must show that the map is surjective. From [4, ??] for every \( \sigma \)-conjugacy class \( O \) in \( W \), there exists a \( \tilde{g} \in G_x(\mathfrak{F}) \) such that \( g^T \) is a maximal \( \mathfrak{f} \)-torus in \( G_x \) and the image of \( \sigma(\tilde{g})^{-1}\tilde{g} \) in \( W \) lies in \( O \). As in Lemma 2.2.3 we can lift \( g^T \) to a maximal unramified torus \( T' \) in \( G \) with \( x \in B(T', k) = A(T', K)^\Gamma \). Since \( x \in A(T', K) \cap A(T, K) \), there exists a \( g \in G(K)_x \) such that \( T' = g^T \). Let \( \tilde{g} \) denote the image of \( g \) in \( G_x(\mathfrak{F}) \). Since \( g^T = \tilde{g}^T \), from [4, ??], the image of \( \sigma(\tilde{g})^{-1}\tilde{g} \) lies in \( O \). Now, from the proof of Lemma 2.2.2, the tori \( \sigma(\tilde{g})^{-1}g^T \) and \( T \) are \( G(K)_x^+ \)-conjugate. Since \( H^1(\Gamma, G(K)_x^+) \) is trivial, we conclude that we may assume that \( \sigma(\tilde{g})^{-1}g^T = T \) Thus, \( \omega(g^T) = O \).

\begin{remark}
We continue to assume that the hypotheses of Theorem 4.0.3 are valid. Suppose \((F_i, T_i) \in I^m\) for \( i = 1, 2 \). The proof of Theorem 4.0.3 yields the following simple criterion for determining if \( C(F_1, T_1) \) and \( C(F_2, T_2) \) lie in the same stable conjugacy class. With out loss of generality, we assume that \( F_1, F_2 \subseteq A(S, k) \). Let \( T_i \) denote the maximal \( \mathfrak{f} \)-torus in \( G_{F_i} \) corresponding to \( T \). Let \( W_i := N_{G_{F_i}}(T_i)(\mathfrak{F})/T_i(\mathfrak{F}) \). Let \( O_i \) be the \( \sigma \)-conjugacy class in \( W_i \) parameterizing the \( G_{F_i}(\mathfrak{f}) \)-conjugacy class of \( T_i(\mathfrak{f}) \). Let \( O_i \) be the \( \sigma \)-conjugacy class in \( W \) \( N_G(T)(K) \) obtained by lifting \( O_i \), into \( N_G(T)(K) \) and then modding out by \( T(K) \). We have \( C(F_1, T_1) \) and \( C(F_2, T_2) \) lie in the same stable conjugacy class if and only if \( O_1 = O_2 \).
\end{remark}

\section*{References}


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