1. Call two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space $V$ are *equivalent* if there exists a real number $C > 0$ such that for all $v \in V$,

$$\frac{1}{C}\|v\|_1 \leq \|v\|_2 \leq C\|v\|_1.$$  

(a) Let $V$ be a vector space. Show that equivalence of norms defines an equivalence relation on the space of all norms on $V$.

(b) Now consider $V \mathbb{R}^n$. Let $\|v\|_1$ be $L^1$ norm on $\mathbb{R}^n$. Let $\|\cdot\|_2$ be another norm. Show that there exists a real number $C > 0$ such that for all $v \in V$,

$$\|v\|_2 \leq C\|v\|_1.$$  

(c) Show that $v \mapsto \|v\|_2, \mathbb{R}^n \mapsto \mathbb{R}$ defines a continuous function on $\mathbb{R}^n$ with respect to the $L^1$-metric.

(d) Show that any two norms on $\mathbb{R}^n$ are equivalent.

2. Extend the notion of uniform continuity to metric spaces.

3. Let $X$ be a compact metric space. Let $C^0(X)$ be the set of continuous real valued functions endowed with the sup norm. Show that $C^0(X)$ is a complete metric space.

4. Recall that the $L^1$-metric on $[0, 1]$ is induced by the norm

$$\|f\|_1 = \int_{[0, 1]} |f(x)| dx$$

where $f$ is a continuous function. As usual, set $d(f, g) = \|f - g\|_1$. This is called the $L^1$ metric. Is $C^0(X)$ with this $L^1$ metric a complete metric space?
5. Let $X$ and $Y$ be metric spaces. A function $f : X \mapsto Y$ is called a \textit{Lipschitz function} if there exists a constant $C > 0$ such that for all $x_1, x_2 \in X$, 
\[ d(f(x_1), f(x_2)) \leq C d(x_1, x_2). \]
We call such $C$ a Lipschitz constant for $f$.

(a) Show that Lipschitz maps are uniformly continuous. Is the converse true?

(b) Let $f_n : X \mapsto Y$ be Lipschitz maps with common Lipschitz constant $C$. Suppose that the $f_n$ converge uniformly to $f$. Is $f$ Lipschitz? What if we only assume that the $f_n$ are Lipschitz?

6. When are differentiable $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ Lipschitz?