Handout 3

1. Let $\mathcal{M}$ be a $\sigma$-algebra. A measure on $\mathcal{M}$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that for any disjoint union of countably many sets we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

We will assume in addition that $\mu(A) < \infty$ for at least one $A \in \mathcal{M}$. Let $X$ be any set, and $\mathcal{M}$ the powerset of $X$, i.e. all subsets of $X$. Show that the following are measures:

(a) For $E \subset X$, let $\mu(E)$ be $\infty$ if $E$ is infinite and the cardinality of $E$ if $E$ is finite. This measure is called the counting measure on $X$.

(b) Fix $p \in X$. For $E \subset X$, let $\mu(E) = 1$ if $p \in E$ and $\mu(E) = 0$ otherwise. This $\mu$ is called the unit mass at $p$ or also the Dirac $\delta$-measure at $p$.

(c) Let $X$ be uncountable. Consider the $\sigma$-algebra $\mathcal{B}$ of all subsets $E \subset X$ such that either $E$ or its complement is at most countable. Let $\mu(E) = 0$ if $E$ is countable. For all other $E \in \mathcal{B}$, let $\mu(E) = 1$.

Note: This property is called countably additive. More strictly one speaks of a positive measure. A countable additive complex valued function on a $\sigma$-algebra is called a complex measure.

2. Check the following properties for a (positive) measure $\mu$:

(a) $\mu(\emptyset) = 0$

(b) for $A, B \in \mathcal{M}$, $A \subset B$ implies $\mu(A) \leq \mu(B)$

(c) for any increasing sequence $A_1 \subset A_2 \subset \ldots A_n \subset \ldots$

$$\mu\left(\bigcup_{i=0}^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n)$$

(d) for a decreasing sequence $A_1 \supset A_2 \supset \ldots \supset A_n \ldots$ with $\mu(A_1) < \infty$:

$$\mu\left(\bigcap_{i=0}^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n)$$