Handout 5

1. A *simple* function on a measurable space $X$ is a measurable function $g : X \mapsto [0, \infty)$ with finite range. Given a measure space $(X, \Sigma, \mu)$ with $\mu(X)$ finite, and $f : X \mapsto \mathbb{R}$ a simple function taking values $p_1, \ldots, p_m$, we defined

$$\int_X f \, d\mu = \sum_{i=1}^{m} p_i \mu(f^{-1}(p_i)).$$

Compare this new procedure for integration with the Riemann integral. What’s different geometrically?

To be more precise, consider $X = [0, 1]$ with the Borel $\sigma$ algebra $\Sigma$. Suppose you have a measure $\mu$ on $\Sigma$ such that $\mu([a, b]) = b - a$. Work out both the new integral and Riemann integral in this case.

2. Let $f : X \mapsto [0, \infty)$ be measurable. Then there exist simple measurable functions $s_n : X \mapsto [0, \infty)$ such that

   (a) $0 \leq s_1 \leq s_2 \leq \ldots \leq f$.
   (b) $s_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$.

3. Let $\mathcal{M}$ be a $\sigma$-algebra on a set $X$, and $\mu$ a (positive) measure on $\mathcal{M}$. If $s : X \to [0, \infty)$ is a simple function of the form $s = \sum \alpha_i \chi_{A_i}$ where $\chi_{A}$ denotes the characteristic function of $A$, then we define for $E \in \mathcal{M}$

$$\int_E s = \sum \alpha_i \mu(E \cap A_i).$$

If $f : X \to [0, \infty)$ is measurable, and $E \in \mathcal{M}$, then we define the *Lebesgue integral* of $f$ over $E$ w.r.t. $\mu$ by

$$\int_E f \, d\mu = \sup \int_E s$$

where the supremum is taken over all simple measurable functions $s$ with $0 \leq s \leq f$.

Check the following:

   (a) If $0 \leq f \leq g$, then $\int_E f \, d\mu \leq \int_E g \, d\mu$. 


(b) If $0 \leq f$ and $A \subset B$ then $\int_A f \, d\mu \leq \int_B g \, d\mu$.

(c) If $0 \leq f$ and $0 \leq c$ is a constant then $\int_A cf \, d\mu = c \int_A f \, d\mu$.

(d) If $f(x) = 0$ for all $x \in E$, then $\int_E f \, d\mu = 0$.

(e) If $0 \leq f$, then $\int_E f \, d\mu = \int_X \chi_E \, f \, d\mu$.

4. Let $X$ be a measurable space, and $f_n : X \to \mathbb{R}$ a sequence of measurable functions. Show that the set of points

$$L := \{x \in X \mid \lim_{n \to \infty} f_n(x) \text{ exists}\}$$

is a measurable subset.