Handout 6

1. Let $A \subset \mathbb{R}$. Consider all possible countable collections $\{I_n\}$ of intervals such that $A \subset \bigcup_n I_n$. For any interval $I$ denote by $l(I)$ the length of $I$. Define the outer measure $\lambda^*(A)$ of $A$ by

$$\lambda^*(A) = \inf \sum l(I_n).$$

Show that $\lambda^*(I) = l(I)$ for any (open or closed) interval $I$.

2. For the outer measure $\lambda^*$ defined above, show that for any countable collection $\{A_n\}$ of subsets of $\mathbb{R}$ that:

$$\lambda^*(\bigcup A_n) \leq \sum \lambda^*(A_n).$$

What is the outer measure of any countable set? Use this to show that $[0, 1]$ is not countable.

3. Is the outer measure $\lambda^*$ countably additive?

4. Show that $\lambda^*$ is translation invariant, i.e. for any subset $A \subset \mathbb{R}$ and any $t \in \mathbb{R}$,

$$\lambda^*(t + A) = \lambda^*(A).$$