Homework 3
for Wednesday, February 7

(1) Let $M$ and $N$ be Riemannian manifolds, with metrics $\langle , \rangle^M$ and $\langle , \rangle^N$, connections $\nabla^M$ and $\nabla^N$ and covariant derivatives $\frac{D^M}{dt}$ and $\frac{D^N}{dt}$ respectively. Let $\phi : M \mapsto N$ an isometry. For $X$ a vector field on $M$, $\phi_*(X)$ fetes the push forward vector field.
   
   (a) Show that $\phi_*(\nabla^M_X Y) = \nabla^N_{\phi_*(X)} \phi_*(Y)$.
   
   (b) Let $c : [a, b] \rightarrow M$ be a differentiable curve. Let $X$ be a vector field along $c$. Let $\phi_*(X)$ denote the vector field along $\phi(c)$ defined by $\phi_*(X)(\phi(c(t))) = D\phi_c(t)(X(c(t)))$.
   
   The show that $\phi_* \left( \frac{D^M}{dt} Y \right) = \frac{D^N}{dt} (\phi_*(Y))$
   
   (c) Let $\phi$ be a geodesic (i.e. a self parallel curve) in $M$. Show that $\phi \circ c$ is a geodesic in $N$.

(2) Let $\mathbb{H}^2$ be the upper half plane $\{(x, y)| x, y \in \mathbb{R}, y > 0\}$ with the Riemannian metric $\langle v, w \rangle_{(x, y)} := \frac{1}{y^2} \langle v, w \rangle_{\text{Eucl}}$.

   (a) Show that the Christoffel symbols are
   
   $\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0$
   
   $\Gamma^2_{11} = \frac{1}{y}$
   
   $\Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}$

   (b) Show that the vertical line $x = 0$ is a geodesic. What is the precise parametrization to give unit speed?

   (c) Show that linear fractional transformations are isometries of $\mathbb{H}^2$. More precisely, write $z = x + iy$ for $(x, y) \in \mathbb{H}^2$. Also let

   $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$

   have determinant 1. Then define the linear fractal transformation by

   $z \mapsto \frac{az + b}{cz + d}$.
Hint: It may help to know that the group of linear fractional transformations is generated by translation $z \mapsto z + b$, scalings $z \mapsto a^2 z$ and the inversion $z \mapsto \frac{1}{z}$. Please assume this if you like.

(d) Then show that the general geodesic is a half circle perpendicular to the boundary of $\mathbb{H}^2$, the $x$-axis. Here we interpret straight vertical lines as half circles of infinite radius. Feel free to use your knowledge from complex analysis here.

(3) Let $\phi : M \mapsto M$ be an isometry of a Riemannian manifold $M$. Let

$$Fix\phi = \{ p \in M | \phi(p) = p \}.$$

(a) Show that $Fix\phi$ is totally geodesic in the following sense: For $p \in Fix\phi$, there exists $\delta > 0$ such that if $d(p, q) < \delta$ and $q \in Fix\phi$, then there is a unique shortest geodesic $\gamma$ from $p$ to $q$, and $\gamma \subset Fix\phi$.

(b) Use this to determine the geodesics on the round sphere (up to parametrization) and for hyperbolic 2-space $\mathbb{H}^2$.

(4) Let $G$ be a Lie group with bi-invariant metric. Show that the geodesics through $1 \in G$ are the same as the one-parameter subgroups.