Homework 4
for Wednesday, February 12

(1) Let \( M \) be a compact Riemannian manifold.
   (a) Show that every non-trivial homotopy class of closed curves contains a shortest
curve \( c \) which then is a geodesic.
   (b) Is this geodesic unique? Prove it or give counterexamples.
   (c) Is the claim from the first part true for non-compact manifolds?

(2) Let \( M \) be a Riemannian manifold. Call a vector field \( X \) on \( M \) Killing if the local flow
generated by \( X \) consists of isometries (more precisely, if for open sets \( U \) on which the
local flow is defined for a uniform time \( t \in (-\varepsilon, \varepsilon) \) the local flow
consists of isometries ).
   (a) Let \( M = \mathbb{R}^n \) with the Euclidean metric. Show that the Killing fields \( X \) with
\( X(0) = 0 \) are given by \( x \mapsto Ax \) where \( A \) is a skew symmetric matrix.
   (b) Let \( p \in M \) and let \( U \) be a normal neighborhood of \( p \). Suppose that \( X(p) = 0 \).
   Show that \( X \) is tangent to the distance spheres \( S_r(p) \) in \( U \).
   (c) Show that push forwards of Killing fields under isometries are Killing fields.
   (d) Show that \( X \) is Killing if and only if \( \langle \nabla Y X, Z \rangle + \langle \nabla Z X, Y \rangle = 0 \) for all vector
fields \( Y, Z \) on \( M \).

(3) Let \( M \) be a connected Riemannian manifold and let \( p \in M \). Let \( \phi : M \to M \) be
an isometry. Show that \( \phi \) is uniquely determined by the value \( \phi(p) \) and derivative
\( D_p(\phi) \) at \( p \).

(4) Let \( M \) be a connected Riemannian manifold, and
   (a) Let \( \phi : M \mapsto M \) a bijection which preserves distances. Show that \( \phi \) has partial
derivatives at every point.
   (b) Show \( \phi \) is differentiable.
   (c) Assuming part (b), show that the isometry group of a compact connected Rie-
mannian manifold is compact.