Homework 9
for Friday, April 11

(1) Suppose $M$ is a simply connected complete manifold of nonpositive curvature. Show that every geodesic triangle satisfies the law of cosines
$$c^2 \geq a^2 + b^2 - 2ab \cos \gamma$$
where $a, b, c$ are the lengths of the sides and $\gamma$ is the angle opposite $a$. Is the assumption that $M$ be simply connected needed?

(2) Suppose $M$ is a simply connected complete manifold of nonpositive curvature.
(a) Let $\gamma$ be a geodesic in $M$. For $p \in M$, show that the function
$$t \mapsto d(p, \gamma(t))$$
is strictly convex.
(b) Let $N \subset M$ be a connected totally geodesic submanifold. Show that for all $p \in M$, there is a unique closest point $\pi(p) \in N$. Show that $\pi : M \mapsto N$ is a distance non-increasing projection.

(3) Let $M$ be a complete Riemannian manifold with sectional curvature $K \leq 1$. Suppose $\gamma_0$ and $\gamma_1$ are distinct geodesics connecting two points $p$ and $q$. Suppose $l(\gamma_0) \leq l(\gamma_1)$. Suppose $\gamma_0$ is homotopic to $\gamma_1$ via a continuous family of differentiable curves $\alpha_s$ for $0 \leq s \leq 1$.
Show that there exists $0 \leq s_0 \leq 1$ such that
$$l(\alpha_{s_0}) \geq 2\pi.$$ 
Hint: There is a long hint in do Carmo, p. 236.

(4) Let $\pi : \tilde{M} \mapsto M$ be a Riemannian submersion. Call $v \in T_{\tilde{p}}\tilde{M}$ horizontal if $v$ is perpendicular to the the fibre of $\pi^{-1}(\pi(p))$. For a vector field $x$ on $M$, define its horizontal lift $\tilde{X}$ on $\tilde{M}$ by requiring that $\tilde{X}(\tilde{p})$ is horizontal for all $\tilde{p} \in \tilde{M}$.
(a) Let $\nabla$ and $\tilde{\nabla}$ denote the Riemannian connections of $M$ and $\tilde{M}$ respectively. Show that
$$\tilde{\nabla}_X \tilde{Y} = (\nabla_X \tilde{Y}) + \frac{1}{2} [\tilde{X}, \tilde{Y}]^v$$
where $x^v$ denotes the vertical (tangent to the fiber) component of a tangent vector $x$ to $\tilde{M}$.
(b) Show that $[\tilde{X}, \tilde{Y}]^v(\tilde{p})$ depends only on $\tilde{X}(\tilde{p})$ and $\tilde{Y}(\tilde{p})$. 