REGULARITY OF CONJUGACIES OF ALGEBRAIC ACTIONS OF ZARISKI DENSE GROUPS

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Abstract. Let α_0 be an affine action of a discrete group Γ on a compact homogeneous space X and α_1 a smooth action of Γ on X which is C^1-close to α_0. We show that under some conditions, every topological conjugacy between α_0 and α_1 is smooth. In particular, our results apply to Zariski dense subgroups of $\mathrm{SL}_d(\mathbb{Z})$ acting on the torus $\mathbb{T}^d$ and Zariski dense subgroups of a simple noncompact Lie group G acting on a compact homogeneous spaces X of G.

Contents

1. Introduction

The investigation of rigidity properties has been at the forefront of research in dynamics in the past two decades. Of particular interest has been the study of higher rank abelian groups and local rigidity of their actions by Hurder, Katok, Lewis, and the last author amongst others. Remarkably, many such

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actions cannot be perturbed at all, in the sense that any C^1-close perturbation is C^∞-conjugate to the original action. Critically, these groups contain higher rank abelian groups. Similar results were found for higher rank semisimple Lie groups and their lattices by Hurder, Lewis, Fisher, Margulis, Qian and others. We refer to [6] for a more extensive survey of these developments.

Smoothness of the conjugacy for these actions came as quite a surprise. Classically, in fact, the stability results of Anosov and later Hirsch, Pugh and Shub guaranteed a continuous conjugacy or orbit equivalence between a single Anosov diffeomorphism or flow and their perturbations [14]. Simple examples however show that such a conjugacy cannot be even C^1 in general.

In the present paper, we investigate similar regularity phenomena for affine actions of a large class of groups. Notably, our results do not require the presence of higher rank subgroups or any assumptions on the structure of the group. In particular they hold for discrete subgroups of rank one semisimple groups. We recall that a group acts *affinely* on a homogeneous space H/Λ for H a Lie group and Λ a discrete subgroup if every element acts by an affine diffeomorphism i.e. one which lifts to a composite of a translation and an automorphism on H. We denote by $\mathrm{Diff}(X)$ the group of C^∞-diffeomorphisms of a space X.

For simplicity let us mention two corollaries of our main theorem in Section 2.

Theorem 1.1. *Let $\Gamma \subset \mathrm{SL}_d(\mathbb{Z})$ for $d \geq 2$ be a finitely generated Zariski dense subgroup in $\mathrm{SL}_d(\mathbb{R})$, and α_0 the associated action on the d-torus $\mathbb{T}^d$. If a perturbation $\alpha_1 : \Gamma \to \mathrm{Diff}(\mathbb{T}^d)$ is sufficiently C^1-close to α_0, then any C^0-conjugacy $\Phi : \mathbb{T}^d \mapsto \mathbb{T}^d$ between α_0 and α_1 is a C^∞-diffeomorphism.*

For $d = 2$, E. Cawley found a $C^{1+\alpha}$-regularity result for Zariski-dense subgroups of $\mathrm{SL}_2(\mathbb{Z})$ acting on the 2-torus in [4] in the early 1990's. Her techniques however are restricted to the 2-torus due to the use of $C^{1+\alpha}$-regularity of stable foliations. Subsequently, the second author obtained a general C^∞-regularity theorem for groups acting on general tori in his thesis [15].

A second application of our main theorem to actions on homogeneous spaces of semisimple groups is novel.

Theorem 1.2. *Let G be a connected simple noncompact Lie group, Λ a cocompact lattice in G, and Γ a finitely generated Zariski dense subgroup of G. Let α_0 be the affine action of Γ on G/Λ. If a C^∞-action α_1 is sufficiently C^1-close to α_0, then any C^0-conjugacy $\Phi : G/\Lambda \mapsto G/\Lambda$ is a C^∞-diffeomorphism.*

Let us note that our techniques are based use certain mixing properties of the actions and do not allow the treatment of actions on general nilmanifolds.

Fisher and Hitchman recently proved a local rigidity theorem for actions of lattices with the Kazhdan property [9]. We recall that an action α is called $C^{k,l}$-rigid if any C^k-close perturbation of the action is C^l-conjugate to α.

Theorem 1.3 (Fisher-Hitchman)**.** *Let Γ be a lattice in a semisimple Lie group without compact factors which satisfies Kazhdan's property. Then any affine action α of Γ is $C^{3,0}$-locally rigid.*

Fisher and Hitchman actually prove this for *quasi-affine* actions, which are extensions of affine actions by isometries. Their technique is based on a type of heat flow. If α does not admit a common neutral direction, then Fisher and Hitchman's proof yields $C^{1,0}$-local rigidity. Using our regularity result, we immediately obtain

Corollary 1.4. *Let G be a simple noncompact Lie group which satisfies Kazhdan's property, Γ a lattice in G, and X a compact homogeneous space of G. Then the affine action of Γ on X is $C^{1,\infty}$-locally rigid.*

Remark 1.5. We can also deduce $C^{1,\infty}$-local rigidity for the action of a Kazhdan lattice Γ, embedded in $\mathrm{SL}_d(\mathbb{Z})$, on the torus $\mathbb{T}^d$ under the assumption that $\Gamma \times \Gamma$ is not contained in the subvarieties $\det([X^\ell, Y] - id) = 0$, $\ell \geq 1$, $\phi(\ell) \leq d^2$, where ϕ is the Euler totient function (see Lemma 4.2). This assumption is needed to construct good pairs in Γ (see Definition 2.1).

Fisher and Hitchman proved $C^{\infty,\infty}$-local rigidity for a more general class of actions of cocompact lattices in the same groups [9]. In particular their approach works on nilmanifolds.

At the heart of our argument lies the investigation of sequences of the form $\gamma^{-n}\delta\gamma^n$ for two hyperbolic elements γ and δ in "general position". Such elements always exist in Zariski-dense groups. The behavior of these sequences is badly divergent in directions transverse to the fast stable direction of γ, and cannot be controlled. However, these sequences do converge along the fast stable manifolds of γ. This is elementary for an affine action. We prove C^1-convergence for the perturbed action. These limiting maps along the fast stable foliation of γ form a rich system which acts transitively along the fast stable leaves under suitable conditions. Moreover, the conjugacy Φ between the actions will also intertwine these limiting maps along fast stables. It follows that Φ has to be C^1 along each of these fast stable manifolds. We prove smoothness in a separate argument.

The proof of C^1-convergence is technically the most difficult piece of the argument. It requires careful estimates which are an adaptation of the proof of Livsic' theorem for cocycles with non-abelian targets.

The use of sequences of the form $\gamma^{-n}\delta\gamma^n$ was introduced by Hitchman in his thesis [15]. His argument relied on the idea that the resulting limit maps along fast stable leaves often exhibit higher rank abelian behavior which could then be used to prove regularity similar to the case of actions by higher rank abelian groups.

Let us comment that our arguments seem to be of rather general nature. In the weakly hyperbolic setting, the hard part in proving local rigidity results lies in getting a C^0-conjugacy. Indeed, the common strategy for most of the known local rigidity results has been to show existence of a C^0-conjugacy and then improve the regularity. Margulis–Qian in higher rank and Fisher-Hitchman for all Kazhdan Lie groups have the most extensive results [20, 10]. The current paper shows regularity under rather general conditions, reducing smooth local rigidity to continuous local rigidity. To pinpoint precisely when local rigidity holds appears difficult. On the one hand, we have the results above for actions of lattices in the Kazhdan rank one groups. On the other hand, Fisher found non-trivial affine deformations of actions of lattices in $SO(n, 1)$ resulting from "bending lattices" [7, 8]. Finally, if the action has isometric directions, even regularity becomes difficult as evidenced even in higher rank by the works of Fisher and Margulis [11] and Fisher and Hitchman [9].

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2. Main result

Let G be a connected Lie group, Λ a cocompact lattice in G, and $X = G/\Lambda$. The group $\mathrm{Aff}(X)$ of affine transformations of X consists of maps of the form

$$f : x \mapsto L_g \circ a(x), \quad x \in X,$$

where L_g denotes the left mutiplication action of $g \in G$ and a is an automorphism of G preserving Λ. Every such map f defines an automorphism Df of $\mathrm{Lie}(G) \simeq T_{e\Lambda}(X)$ given by

$$Df := \mathrm{Ad}(g) \circ D(a)_e.$$

We denote by W_f^{min} the sum of the generalized eigenspaces of Df with eigenvalues of minimal modulus and by $P_f^{min} : \mathrm{Lie}(G) \to W_f^{min}$ the projection map along the other generalized eigenspaces.

Definition 2.1. *We call a pair* $f, g \in \mathrm{Aff}(X)$ good *if the following conditions are satisfied:*

(i) *The map* Df *is partially hyperbolic.*
(ii) *The map* $Df : W_f^{min} \to W_f^{min}$ *is semisimple.*
(iii) *The map* $P_f^{min} \circ Dg : W_f^{min} \to W_f^{min}$ *is nondegenerate.*
(iv) *For every subsequence* $\{n_i\}$ *, the sequence* $\{f^{-n_i} g f^{n_i}(x)\}$ *is dense in* X *for* x *in a set of full measure.*

If for $f \in \mathrm{Aff}(X)$*, there exists* $g \in \mathrm{Aff}(X)$ *so that the pair* f, g *is good, we say* f *is a* good *mapping.*

Remark 2.2. In the case when the map $Df : W_f^{min} \to W_f^{min}$ does not have a rotation component of infinite order (e.g., when $\dim W_f^{min} = 1$), it suffices to assume that the sequence $\{f^{-n} g f^n(x)\}$ is dense in X for x in a set of full measure. In general, we have to pass to a subsequence to guarantee that the maps $f^{-n} g f^n$ converge along the fast stable leaves as $n \to \infty$ (see Proposition 3.13).

The theorems stated in the introduction will be deduced from the following general result:

Main Theorem. *Let* Γ *be a finitely generated discrete group and* $\alpha_0 : \Gamma \to \mathrm{Aff}(X)$ *an affine action of* Γ *such that*

- $(D\alpha_0)(\Gamma)$ *acts irreducibly on* $\mathrm{Lie}(G)$*,*
- $\alpha_0(\Gamma)$ *contains a good pair.*

Let $\alpha_1 : \Gamma \to \mathrm{Diff}(X)$ *be a* C^∞*-action of* Γ *which is sufficiently* C^1*-close to* α_0*. Then every homeomorphism* $\Phi : X \to X$ *satisfying*

$$\Phi \circ \alpha_0(\gamma) = \alpha_1(\gamma) \circ \Phi \quad \textit{for all } \gamma \in \Gamma$$

is a C^∞*-diffeomorphism.*

Remark 2.3. Irreducibility of the action of Γ on $\mathrm{Lie}(G)$ is used in the following places:

- In Section 3.1, to deduce weak hyperbolicity (see (1)),
- In Section 3.2, to construct essential sets (see Lemma 3.9),
- In Section 3.5, to deduce that Φ is C^∞ from smoothness on the fast stable leaves (see (41)).

It is clear from the argument in Section 3 that irreducibility can be replaced by the following condition: there exists good $f_0 \in \alpha_0(\Gamma)$ such that

$$\sum_{g \in \alpha_0(\Gamma)} (Dg) W_{f_0}^{min} = \mathrm{Lie}(G) \quad \text{and} \quad \bigcap_{g \in \alpha_0(\Gamma)} (Dg) W_{f_0}^{max} = 0$$

where $W_{f_0}^{max}$ is the sum of generalized eigenspaces complementary to $W_{f_0}^{min}$.

Without the irreducibility assumption, one can still prove that the map Φ is C^∞ restricted to the fast stable leaves of good $f_0 \in \mathrm{Aff}(X)$ (see Theorem 3.17 below).

Remark 2.4. The proof of the Main Theorem also applies to perturbations with finite order of smoothness. One can show that for a C^l-action $\alpha_1 : \Gamma \to \mathrm{Diff}^l(X)$, the conjugacy map Φ is C^l on fast stable manifolds of good elements, and hence by [5, Theorem 3] or [23, Theorem 1.1], Φ is in the Sobolev space of regularity l.

Existence of good pairs for some classes of affine actions will be proved in Section 4. In particular, Theorem 1.1 follows from the Main Theorem and Proposition 4.1, and Theorem 1.2 follows from the Main Theorem and Proposition 4.4.

Outline of the proof of the Main Theorem. Irreducibility of Γ-action and property (i) of a good pair are used to prove that Φ is bi-Hölder (Section 3.1). Next, irreducibility of the Γ-action and property (ii) of a good pair are used to show that Φ maps fast stable manifolds to fast stable manifolds (Section 3.2). Property (ii) is also used to show that a subsequence of maps $f^{-n} g f^n$ restricted to fast stable manifolds is precompact in the C^0-topology and, in fact, in the C^1-topology (Section 3.3). Then one utilizes property (iii) of a good pair to deduce that the limits of these maps are homeomorphisms and property (iv) of a good pair to deduce that these limits generate transitive C^1-action on fast stable manifolds. Using that Φ is a conjugacy between the constructed C^1-actions, we show that Φ is C^1 along the fast stable leaves (Section 3.4). A more elaborate argument shows that Φ is C^∞ along fast stable leaves: when the fast stable leaves are one-dimensional, this argument is based on the nonstationary Sternberg linearization, developed in [17, 13, 12], and in higher-dimensional case, the argument uses Shefel's theorem [25] on smoothness of conformal maps. Finally, we conclude that Φ is C^∞ on X by a standard argument, which uses irreducibility of the Γ-action (Section 3.5). □

3. Proof of the Main Theorem

We continue with the notation that $X = G/\Lambda$ is a compact quotient of a connected Lie group G by a discrete subgroup $\Lambda \subset G$.

3.1. C^0 **implies Hölder.** In this section, we will prove that the conjugacy map $\Phi : X \to X$ in the Main Theorem is bi-Hölder. The proof is similar to Proposition 5.7 of [20]. As they do not show that their map is Hölder, and also use somewhat different hypotheses, we will give a proof here for simplicity.

Following [20], we say that a C^1-action α of a discrete group Γ on a compact manifold M is *weakly hyperbolic* when there is a choice of finitely many elements $\gamma_1, \dots, \gamma_k$ in Γ such that each diffeomorphism $\alpha(\gamma_i)$ is partially hyperbolic and, for each point $x \in M$,

$$\sum_{i=1}^{k} T_x W^s_{\alpha(\gamma_i)}(x) = T_x M, \tag{1}$$

where $W^s_{\alpha(\gamma_i)}(x)$ denotes the stable manifold of $\alpha(\gamma_i)$ through x.

Theorem 3.1. *Let Γ be a finitely generated discrete group, $\alpha_0 : \Gamma \to \mathrm{Aff}(X)$ be an affine weakly hyperbolic action, and $\alpha_1 : \Gamma \to \mathrm{Diff}^1(X)$ a smooth action which is sufficiently C^1-close to α_0. Then every homeomorphism $\Phi : X \to X$ such that*

$$\Phi \circ \alpha_0(\gamma) = \alpha_1(\gamma) \circ \Phi \quad \textit{for all } \gamma \in \Gamma$$

is bi-Hölder.

The proof is divided into several lemmas.

Lemma 3.2. *Let $f_1, \dots, f_k$ be partially hyperbolic diffeomorphisms of X such that*

$$\sum_{i=1}^{k} T_x W^s_{f_i}(x) = T_x M \quad \textit{for all } x \in X,$$

and $g_1, \dots, g_k$ are C^1-close C^1-diffeomorphisms. Then g_i's are partially hyperbolic and

$$\sum_{i=1}^{k} T_x W^s_{g_i}(x) = T_x M \quad \textit{for all } x \in X.$$

Lemma 3.2 follows from stability of partial hyperbolicity under perturbations (see, for example, [21, Lemma 3.5]).

Lemma 3.3. *Let Φ be a continuous conjugacy between two partially hyperbolic diffeomorphisms of a compact manifold. Then Φ is bi-Hölder continuous along the stable manifolds of these mappings.*

Lemma 3.3 follows from the standard argument as in [16, Theorem 19.1.2].

Lemma 3.4. *Let $\alpha : \Gamma \to \mathrm{Diff}^1(X)$ be a smooth weakly hyperbolic action and $\gamma_1, \dots, \gamma_k \in \Gamma$ satisfy (1). Then there exist $c, \epsilon > 0$ such that for every $x, y \in X$ satisfying $d(x, y) < \epsilon$, there exists a path ℓ from x to y which consists of $2k$ pieces contained in stable manifolds of $\alpha(\gamma_1), \dots, \alpha(\gamma_k)$, and $L(\ell) \leq c\, d(x, y)$.*

Proof. We will use an argument similar to [24, Lemma 3.1].

There exists a family of continuous unit vector fields $v_1, \dots, v_d$ such that the stable distribution of $\alpha(\gamma_i)$ is the span of $v_{d_{i-1}}, \dots, v_{d_i - 1}$ for some $1 = d_0 \le d_1 \le \cdots \le d_k = d+1$. Let $\delta > 0$. There exists $\delta' > 0$ such that $d(u, w) < \delta'$ implies that $d(v_i(u), v_i(w)) < \delta$ for all i. By [19, Corollary 4.5], for every $x \in X$, there exists $\epsilon(x) > 0$ such that every $y \in B_{\epsilon(x)}(x)$ can be connected to x by a path ℓ of length at most $\delta'/2$, and for some $0 = t_0 \le t_1 \le \cdots \le t_k = L(\ell)$, we have $\ell'(t) = v_i(\ell(t))$ when $t \in [t_{i-1}, t_i)$. Let $\epsilon > 0$ be the Lebesgue number of the cover $\{B_{\epsilon(x)}(x)\}$. Then every $y_1, y_2 \in X$ such that $d(y_1, y_2) < \epsilon$ are connected by a path ℓ which consists of $2k$ pieces tangent to v_j's and $L(\ell) < \delta'$. To estimate the distance $d(y_1, y_2)$, we may assume, without loss of generality, that we work in an open neighborhood of $\mathbb{R}^d$ equipped with the standard metric. By the triangle inequality,

$$\begin{aligned}
\|y_1 - y_2\| &= \left\| \sum_j \int_{t_{j-1}}^{t_j} v_{i_j}(\ell(t)) dt \right\| \\
&\ge \left\| \sum_j (t_j - t_{j-1}) v_{i_j}(y_1) \right\| - \sum_j \int_{t_{j-1}}^{t_j} \|v_{i_j}(\ell(t)) - v_{i_j}(y_1)\| dt \\
&\ge (c - \delta) L(\ell)
\end{aligned}$$

for some fixed $c > 0$ independent of δ and y_1. Taking δ sufficiently small, this implies the estimate for $L(\ell)$. Since the stable distributions are uniquely integrable, $\ell([t_{j-1}, t_j))$ is contained in the stable manifold of $\alpha(\gamma_{i_j})$. □

Proof of Theorem 3.1. Let $\gamma_1, \dots, \gamma_k \in \Gamma$ be elements satisfying (1). By Lemma 3.3, the map Φ is bi-Hölder restricted to the stable manifolds of $\alpha_0(\gamma_i)$'s. By Lemma 3.4, for sufficiently close $x, y \in X$, there exist points $x_0 = x, x_1, \dots, x_{2k} = y$ such that x_{j-1} and x_j are on the same stable manifold of some $\alpha_0(\gamma_{i_j})$, and $d(x_{j-1}, x_j) \le c\, d(x, y)$. Then

$$\begin{aligned}
d(\Phi(x), \Phi(y)) &\le \sum_{j=1}^{2k} d(\Phi(x_{j-1}), \Phi(x_j)) \le \sum_{j=1}^{2k} c_j d(x_{j-1}, x_j)^{\theta_j} \\
&\le \left(\sum_{j=1}^{2k} c_j c^{\theta_j} \right) d(x, y)^{\theta}
\end{aligned}$$

where $\theta = \min \theta_j$.

By Lemma 3.2, the action α_1 is also weakly hyperbolic, Then the proof that Φ^{-1} is Hölder follows the same argument. □

3.2. **Invariance of fast stable manifolds.** Let $f \in \mathrm{Diff}(X)$, and the tangent bundle TX has continuous f-invariant splitting

$$TX = E^- \oplus E^+ \tag{2}$$

such that for some $\lambda \in (0,1)$ and $\mu > \lambda$,[1]

$$\begin{aligned} \|D(f^n)_x v\| &\ll \lambda^n \|v\| \quad \text{for all } n \geq 0,\ x \in X, \text{ and } v \in E^-_x, \\ \|D(f^n)_x v\| &\gg \mu^n \|v\| \quad \text{for all } n \geq 0,\ x \in X, \text{ and } v \in E^+_x. \end{aligned} \tag{3}$$

We recall (see, for example, [21, Theorem 4.1]) that the distribution E^- is integrable to the *fast stable* foliation $\{W^{fs}_f(x)\}_{x \in X}$, and this foliation is Hölder continuous with C^∞-leaves. We denote by d^{fs} the induced metrics on the leaves of this foliation. For $\rho > \lambda$ and $x, y \in X$ such that $y \in W^{fs}_f(x)$,

$$d^{fs}(f^n(x), f^n(y)) \ll \rho^n d^{fs}(x, y).$$

There exists $\epsilon_0 > 0$ such that for every $z, w \in X$ satisfying $w \in W^{fs}_f(z)$ and $d^{fs}(z, w) < \epsilon_0$, we have

$$d^{fs}(z, w) \ll d(z, w) \leq d^{fs}(z, w). \tag{4}$$

Let $f_0 \in \mathrm{Aff}(X)$ be such that Df_0 is partially hyperbolic, and $\lambda_0 < \mu_0$ denote the least two absolute values of the eigenvalues of Df_0. If $f \in \mathrm{Diff}(X)$ is a C^1-small perturbation of f_0, then we have a splitting as above with $\lambda = \lambda_0 + \epsilon$ and $\mu = \mu_0 - \epsilon$ for some small $\epsilon > 0$, depending on $d_{C^1}(f, f_0)$ (see [21, Lemma 3.5]). The fast stable manifolds $W^{fs}_f(x)$ are defined with respect to this splitting. Note that

$$W^{fs}_{f_0}(x) = \exp(W^{min}_{f_0})x$$

where exp is the Lie exponential map, and $W^{min}_{f_0}$ is defined as on page 4.

The aim of this section is to prove the following theorem.

Theorem 3.5. *Let $\alpha_0 : \Gamma \to \mathrm{Aff}(X)$ and $\alpha_1 : \Gamma \to \mathrm{Diff}(X)$ be C^1-close actions of a finitely generated discrete group Γ, and let $\Phi : X \to X$ be a homeomorphism such that*

$$\Phi \circ \alpha_0(\gamma) = \alpha_1(\gamma) \circ \Phi \quad \textit{for all } \gamma \in \Gamma.$$

Assume that $(D\alpha_0)(\Gamma)$ acts irreducibly on $\mathrm{Lie}(G)$. Then for every partially hyperbolic $f_0 := \alpha_0(\gamma)$ and $f := \alpha_1(\gamma)$, $\gamma \in \Gamma$, such that Df_0 is semisimple on $W^{min}_{f_0}$,

$$\Phi(W^{fs}_{f_0}(z)) = W^{fs}_f(\Phi(z)) \quad \textit{for all } z \in X.$$

[1] The notation $A \ll B$ means that there exists $c > 0$, independent of other parameters, such that $A \leq c\,B$.

Moreover, the map Φ is bi-Hölder with respect to the induced metrics on fast stable leaves of f_0 and f.

Let us start with some preliminary reductions. We will prove that

$$\Phi^{-1}(W_f^{fs}(z)) \subset W_{f_0}^{fs}(\Phi^{-1}(z)) \quad \text{for all } z \in X. \tag{5}$$

This also implies that the equality. Indeed, it follows from (5) that every leaf $W_{f_0}^{fs}(\Phi^{-1}(z))$ is a disjoint union of sets of the form $\Phi^{-1}(W_f^{fs}(y))$ for some $y \in X$. By [21, Lemma 3.5], the fast stable leaves of f_0 and f have the same dimension. Hence, by the invariance of domain, every set $\Phi^{-1}(W_f^{fs}(y))$ is open in $W_{f_0}^{fs}(\Phi^{-1}(z))$. Since $W_{f_0}^{fs}(\Phi^{-1}(z))$ is connected, we deduce that

$$\Phi^{-1}(W_f^{fs}(z)) = W_{f_0}^{fs}(\Phi^{-1}(z)).$$

Let

$$\mathcal{S}_{\epsilon'}(x) = \{\Phi^{-1}(z) :\ z \in W_f^{fs}(\Phi(x)),\ d^{fs}(z, \Phi(x)) < \epsilon'\}. \tag{6}$$

We will show that there exists $\epsilon' \in (0, \epsilon_0)$ such that for every $x \in X$,

$$\mathcal{S}_{\epsilon'}(x) \subset W_{f_0}^{fs}(x).$$

This will imply the theorem.

First, we observe the following property of points lying on the same fast stable leaf for affine actions:

Proposition 3.6. *Let $f_0, g_0 \in \mathrm{Aff}(X)$ be such that $(Df_0)|_{W_{f_0}^{min}}$ is semisimple. Then there exists $c > 0$ such that for every $z, w \in X$ satisfying $w \in W_{f_0}^{fs}(z)$ and $n \geq k \geq 0$,*

$$d(f_0^{-k} g_0 f_0^n(z), f_0^{-k} g_0 f_0^n(w)) \leq c\, \lambda_0^{n-k} d^{fs}(z, w)$$

where λ_0 is the least absolute value of the eigenvalues of Df_0.

Proof. It suffices to prove the proposition when $d^{fs}(z, w)$ small. Write $w = \exp(v)z$ for $v \in W_{f_0}^{min}$. Then

$$w = \exp(D(f_0^{-k} g_0 f_0^n)v) f_0^{-k} g_0 f_0^n(z),$$

and it suffices to show that for a norm on $\mathrm{Lie}(G)$,

$$\|D(f_0^{-k} g_0 f_0^n)v\| \ll \lambda_0^{n-k} \|v\|,$$

which is easy to check. □

A similar but weaker property also holds for small perturbations of affine actions:

Proposition 3.7. *Let $f_0 \in \mathrm{Aff}(X)$, $g \in \mathrm{Diff}(X)$, and $\nu > 1$. Then there exists $c > 0$ such that for any sufficiently C^1-small perturbations $f \in \mathrm{Diff}(X)$ of f_0, $z, w \in X$ satisfying $w \in W_f^{fs}(z)$, and $n \geq 0$,*

$$d(f^{-n}gf^n(z), f^{-n}gf^n(w)) \leq c\,\nu^n d^{fs}(z,w). \tag{7}$$

Proof. Let λ_0 denote the least absolute value of the eigenvalues of Df_0. Take $\lambda_- < \lambda_0 < \lambda_+$ such that $\frac{\lambda_+}{\lambda_-} < \nu$. For f sufficiently C^1-close to f_0, we have

$$\|D(f^{-n})_u\| \ll \lambda_-^{-n} \quad \text{for all } u \in X \text{ and } n \geq 0,$$

and

$$\left\| D(f^n)|_{T_u(W_f^{fs}(u))} \right\| \ll \lambda_+^n \quad \text{for all } u \in X \text{ and } n \geq 0.$$

This implies that

$$\left\| D(f^{-n}gf^n)|_{T_u(W_f^{fs}(u))} \right\| \ll \left(\frac{\lambda_+}{\lambda_-}\right)^n \quad \text{for all } u \in X \text{ and } n \geq 0.$$

Let ℓ be a smooth curve in $W_f^{fs}(z)$ from z to w such that $L(\ell) = d^{fs}(z,w)$. Then

$$L(f^{-n}gf^n(\ell)) \ll \left(\frac{\lambda_+}{\lambda_-}\right)^n L(\ell) < \nu^n d^{fs}(z,w)$$

for all $n \geq 0$. This proves the proposition. □

It turns out that property (7) characterizes points lying on the same fast stable leaves. This observation is crucial for the proof of Theorem 3.5 and is the main point of Theorem 3.10 below. Since the proof of Theorem 3.10 is quite involved, we first present its linear analogue – Proposition 3.8. Although the argument in the proof of Theorem 3.10 follows the same idea, it requires more delicate quantitative estimates because we have to work in injectivity neighborhoods of the exponential map.

Let $A \in \mathrm{GL}_l(\mathbb{R})$. We denote by $\lambda_1 < \cdots < \lambda_d$ be the absolute values of the eigenvalues of A, and P_i denote the projection to the sum of the generalized eigenspaces of A corresponding to λ_i along the other eigenspaces.

Proposition 3.8. *Let $B_1, \ldots, B_k \in \mathrm{GL}_l(\mathbb{R})$ be such that for some $\eta > 0$*

$$\max_k \|P_1 B_k v\| > \eta \|v\|, \quad v \in \mathbb{R}^l. \tag{8}$$

Then there exists $\nu > 1$ such that

$$W_A^{min} = \{v : \max_k \|A^{-n} B_k A^n\, v\| = O(\nu^n) \quad \text{as } n \to \infty\}.$$

Proof. For every small $\rho > 0$ there exists a norm on $\mathrm{Lie}(G)$ (see [16, Proposition 1.2.2]) such that $\|v_1 + v_2\| = \|v_1\| + \|v_2\|$ for v_1 and v_2 in different generalized eigenspaces and

$$(\lambda_i - \rho)\|v\| \le \|Av\| \le (\lambda_i + \rho)\|v\|, \quad v \in \mathrm{im}(P_i).$$

The parameter ρ is fixed, but has to be chosen sufficiently small so that

$$(\lambda_1 - \rho)^{-1}(\lambda_1 + \rho) < \min_{i>1}(\lambda_1 + \rho)^{-1}(\lambda_i - \rho).$$

It follows from (8) that

$$\begin{aligned}\max_k \|A^{-n}B_kA^n v\| &\ge \max_k(\lambda_1+\rho)^{-n}\|P_1B_kA^nv\| \\ &\ge (\lambda_1+\rho)^{-n}\eta\|A^nv\| \\ &\ge (\lambda_1+\rho)^{-n}\eta\sum_i(\lambda_i-\rho)^n\|P_iv\|.\end{aligned}$$

We take $\nu > 1$ such that

$$\nu < (\lambda_1 + \rho)^{-1}(\lambda_i - \rho) \text{ for } i > 1 \text{ and } \nu > (\lambda_1 - \rho)^{-1}(\lambda_1 + \rho).$$

Then $\max_k \|A^{-n}B_kA^nv\| = O(\nu^n)$ implies that $P_iv = 0$ for $i > 1$. Also, for $v \in W_A^{min}$,

$$\max_k \|A^{-n}B_kA^nv\| \le (\lambda_1 - \rho)^{-n}\left(\max_k \|B_k\|\right)(\lambda_1 + \rho)^n = O(\nu^n).$$

This proves the proposition. □

Proposition 3.7 and (4) imply that uniformly on $z, w \in X$, satisfying $w \in W_f^{fs}(z)$ and $d^{fs}(z,w) < \epsilon_0$, and $n \ge 0$, we have

$$d(f^{-n}gf^n(z), f^{-n}gf^n(w)) \ll \nu^n d(z,w).$$

Now we take $g = \alpha_1(\delta)$ and $g_0 = \alpha_0(\delta)$ for some $\delta \in \Gamma$. Since the action of Γ on $\mathrm{Lie}(G)$ is irreducible, α_0 is weakly hyperbolic. Hence, by Theorem 3.1, the conjugacy map Φ and its inverse are Hölder with some exponent $\theta > 0$. It follows that uniformly on $x, y \in X$, satisfying $y \in \mathcal{S}_{\epsilon'}(x)$, and $n \ge 0$,

$$\begin{aligned}d(f_0^{-n}g_0f_0^n(x), f_0^{-n}g_0f_0^n(y)) &\ll d(f^{-n}gf^n(\Phi(x)), f^{-n}gf^n(\Phi(y)))^\theta \\ &\ll \nu^{\theta n}d(\Phi(x),\Phi(y))^\theta \\ &\ll \nu^{\theta n}d(x,y)^{\theta^2}.\end{aligned} \tag{9}$$

Let $\lambda_1 < \cdots < \lambda_d$ be the absolute values of the eigenvalues of Df_0 and P_i denote the projection from $\mathrm{Lie}(G)$ to the sum of the generalized eigenspaces of Df_0 corresponding to λ_i along the other generalized eigenspaces.

We say that a set $\{g_1, \dots, g_l\} \subset \mathrm{Aff}(X)$ is *essential* for f_0 if for some $\eta > 0$ and every $v \in \mathrm{Lie}(G)$,

$$\max_k \|P_1(Dg_k)v\| > \eta\|v\|.$$

Note this definition does not depend on a choice of the norm. Existence of essential sets follows from the following lemma:

Lemma 3.9. *A set $g_1, \dots, g_l \in \mathrm{Aff}(X)$ is essential if and only if*

$$\bigcap_{k=1}^{l} (Dg_i)^{-1}\ker(P_1) = 0. \tag{10}$$

In particular, every subgroup $\Gamma \subset \mathrm{Aff}(X)$ such that $D\Gamma$ acts irreducibly on $\mathrm{Lie}(G)$ contains an essential set.

Proof. Since the map

$$v \mapsto (P_1(Dg_k)v : \ k = 1, \dots l) : \mathrm{Lie}(G) \to \mathrm{Lie}(G)^l$$

is injective when (10) holds, one can take

$$\eta = \min\{\max_k \|P_1(Dg_k)v\| : \|v\| = 1\} > 0.$$

The converse is also clear.

To prove the second claim, we observe that there exists a subset $\{g_1, \dots, g_l\} \subset \Gamma$ such that

$$\bigcap_{k=1}^{l} (Dg_k)^{-1}\ker(P_1) = \bigcap_{g\in\Gamma} (Dg)^{-1}\ker(P_1),$$

and this space is zero by irreducibility. □

The following theorem is the main ingredient of the proof of Theorem 3.5:

Theorem 3.10. *There exists $\nu = \nu(\kappa, f_0) > 1$ such that given $a, \kappa > 0$, $f_0 \in \mathrm{Aff}(X)$ such that Df_0 is semisimple on $W^{min}_{f_0}$, an essential set $g_1, \dots, g_l \in \alpha_0(\Gamma)$, and a family of subsets $\mathcal{L}_\epsilon(x)$, $x \in X$, of X that satisfy*

(i) $x \in \mathcal{L}_\epsilon(x) \subset B_\epsilon(x)$,
(ii) $f_0^{-1}(\mathcal{L}_\epsilon(x)) \supset \mathcal{L}_\epsilon(f_0^{-1}(x))$,
(iii) *for every $y \in \mathcal{L}_\epsilon(x)$ and $n \geq 0$,*

$$\max_k d(f_0^{-n} g_k f_0^n(x), f_0^{-n} g_k f_0^n(y)) \leq a\nu^n d(x, y)^\kappa, \tag{11}$$

one can choose $\epsilon > 0$ such that

$$\mathcal{L}_\epsilon(x) \subset W^{fs}_{f_0}(x) \quad \textit{for every } x \in X.$$

Outline of the proof of Theorem 3.10. We first observe that the sets $\mathcal{L}_\epsilon(x)$ lie in "cones" around $W^{fs}_{f_0}(x)$ where the size of the cones is controlled by ν and can be made sufficiently small (Lemma 3.11). Note that this argument is analogous to the proof of Proposition 3.8, but we can only derive a weaker conclusion because one has to work in injectivity neighborhoods of the exponential map. In the next step, we show that applying the map f_0^{-1}, the size of the cones can be made arbitrary small (Lemma 3.12). This implies the theorem. □

We fix a norm on Lie(G), depending on parameter $\rho > 0$, as in the proof of Proposition 3.8 with $A = Df_0$. The parameter ρ has to be chosen sufficiently small. It controls the size of the cone in Lemma 3.11. We always take $\rho > 0$ so that

$$\begin{aligned} &\lambda_i < \lambda_j - \rho \quad \text{when } \lambda_i < \lambda_j, \\ &\lambda_i - \rho > 1 \quad \text{when } \lambda_i > 1, \\ &\lambda_i + \rho < 1 \quad \text{when } \lambda_i < 1. \end{aligned}$$

Note that since $(Df_0)|_{W^{min}_{f_0}}$ is semisimple, we also have

$$\|(Df_0)v\| = \lambda_1 \|v\|, \quad v \in \mathrm{im}(P_1),$$

and

$$\|(Df_0)^{-n}\| \le \lambda_1^{-n}.$$

By the assumption on g_k's, there exists $\eta > 0$ such that

$$\max_k \|P_1(Dg_k)v\| > \eta\|v\|, \quad v \in \mathrm{Lie}(G). \tag{12}$$

Let $\mu_i = \lambda_1^{-1}(\lambda_i + \rho)$ and $\sigma_i = \frac{\log \mu_i}{\log \mu_d}$. For $v \in \mathrm{Lie}(G)$, we define

$$N(v) = \max_{i>1} \left\{ \|P_i v\|^{\sigma_i^{-1}} \right\}.$$

For $\beta, s > 0$, we define

$$C(\beta, s) = \{v \in \mathrm{Lie}(G) :\, N(v) \le \beta \|v\|^s\}.$$

Lemma 3.11. *There exist $\epsilon, \beta > 0$ such that for every $x, y \in X$ satisfying $d(x,y) < \epsilon$ and (11),*

$$y \in \exp(C(\beta, s))x.$$

where $s = s(\nu, \rho, \kappa, f_0) > 0$ is such that $s \to \infty$ as $\nu \to 1^+$ and $\rho \to 0^+$.

Proof. Let $c_1 = \max_k \|Dg_k\|$.

There exist $\delta_0 > 0$ and $c_0 > 1$ such that for every $x \in X$ and $v \in \mathrm{Lie}(G)$ satisfying $\|v\| < \delta_0$, we have

$$c_0^{-1}\|v\| \le d(x, \exp(v)x) \le c_0 \|v\|. \tag{13}$$

Let $b > 0$ such that $\sum_{j>1} b^{\sigma_j} = \delta_0/(2c_1)$. We choose $\epsilon > 0$ so that $d(x, y) < \epsilon$ implies that $y = \exp(v)x$ where

$$N(v) < \min\{1, b\} \quad \text{and} \quad \|v\| < \min\{\delta_0, \delta_0/2c_1\}.$$

Assuming that the claim fails, we will show that there exists $n \geq 0$ such that

$$ac_0^{\kappa+1}\nu^n\|v\|^\kappa < \max_k \|D(f_0^{-n} g_k f_0^n)v\| < \delta_0. \tag{14}$$

Since

$$d(f_0^{-n} g_k f_0^n(x), f_0^{-n} g_k f_0^n(y)) = d(f_0^{-n} g_k f_0^n(x), \exp(D(f_0^{-n} g_k f_0^n)v) f_0^{-n} g_k f_0^n(x)),$$

we deduce from (13) and (14) that

$$ac_0^{\kappa+1}\nu^n(c_0^{-1}\, d(x, y))^\kappa < \max_k c_0\, d(f_0^{-n} g_k f_0^n(x), f_0^{-n} g_k f_0^n(y)),$$

which contradicts (11).

To obtain the upper estimate in (14), we observe that

$$\begin{aligned}\max_k \|D(f_0^{-n} g_k f_0^n)v\| &\leq \lambda_1^{-n} \max_k \|D(g_k f_0^n)v\| \leq \lambda_1^{-n} c_1 \|D(f_0^n)v\| \\ &\leq c_1\|P_1 v\| + \lambda_1^{-n} c_1 \sum_{j>1} (\lambda_j + \rho)^n \|P_j v\| \\ &\leq c_1\|v\| + c_1 \sum_{j>1} \mu_j^n \|P_j v\|.\end{aligned}$$

We choose $n \geq 0$ so that

$$\mu_d^{-1} \frac{b}{N(v)} < \mu_d^n \leq \frac{b}{N(v)}. \tag{15}$$

Then

$$\mu_d^{-\sigma_j} \frac{b^{\sigma_j}}{N(v)^{\sigma_j}} < \mu_j^n \leq \frac{b^{\sigma_j}}{N(v)^{\sigma_j}}$$

and

$$\max_k \|D(f_0^{-n} g_k f_0^n)v\| \leq c_1\|v\| + c_1 \sum_{j>1} b^{\sigma_j} \frac{\|P_j v\|}{N(v)^{\sigma_j}} < \delta_0.$$

The lower estimate in (14) is proved similarly using that $g_1, \dots, g_l$ is essential (see (12)). Let $\gamma_j > 0$ be such that $\lambda_1^{-1}(\lambda_j - \rho) = \mu_j^{1-\gamma_j}$. We have

$$\begin{aligned}\max_k \|D(f_0^{-n} g_k f_0^n)v\| &\geq \max_k \lambda_1^{-n} \|P_1 D(g_k f_0^n)v\| \geq \lambda_1^{-n}\eta \|D(f_0^n)v\| \\ &\geq \lambda_1^{-n}\eta \left(\lambda_1^n \|P_1 v\| + \sum_{j>1} (\lambda_j - \rho)^n \|P_j v\| \right) \\ &\geq \eta \sum_{j>1} \mu_j^{n(1-\gamma_j)} \|P_j v\| \\ &\geq \eta \sum_{j>1} (\mu_d^{-1} b)^{\sigma_j(1-\gamma_j)} N(v)^{\sigma_j\gamma_j} \frac{\|P_j v\|}{N(v)^{\sigma_j}} \\ &\geq \eta (\mu_d^{-1} b)^{\sigma_{j_0}(1-\gamma_{j_0})} N(v)^{\sigma_{j_0}\gamma_{j_0}}.\end{aligned}$$

where $j_0 > 1$ is such that $\|P_{j_0} v\|^{1/\sigma_{j_0}} = N(v)$. This implies that

$$\max_k \|D(f_0^{-n} g_k f_0^n)v\| \geq \min_{j>1} \eta (\mu_d^{-1} b)^{\sigma_j(1-\gamma_j)} N(v)^{\sigma_j\gamma_j}.$$

Let $\omega = \frac{\log \nu}{\log \mu_d}$. It follows from (15) that the first inequality in (14) is satisfied provided that

$$a c_0^{\kappa+1} N(v)^{-\omega} b^{\omega} \|v\|^{\kappa} < \min_{j>1} \eta (\mu_d^{-1} b)^{\sigma_j(1-\gamma_j)} N(v)^{\sigma_j\gamma_j}.$$

Since this gives a contradiction, we deduce that

$$a c_0^{\kappa+1} b^{\omega} \|v\|^{\kappa} \geq \min_{j>1} \eta (\mu_d^{-1} b)^{\sigma_j(1-\gamma_j)} N(v)^{\omega+\sigma_j\gamma_j}.$$

Hence,

$$N(v) \leq \beta \|v\|^s$$

with explicit $\beta > 0$ and $s = \kappa/(\omega + \max_{j>1}(\sigma_j\gamma_j))$. Clearly, $s \to \infty$ as $\nu \to 1^+$ and $\rho \to 0^+$. This completes the proof. □

For $i = 1, \dots, d$ and $\delta, \beta, s > 0$, we define

$$C_\delta^i(\beta, s) = \{v \in \mathrm{Lie}(G) : \|v\| < \delta;\ \|P_i v\|^{\sigma_i^{-1}} \leq \beta \|v\|^s;\ P_j v = 0, j > i\}.$$

Lemma 3.12. *For every $\delta, \beta, s > 0$,*

$$(Df_0)^{-1}(C_\delta^i(\beta, s)) \subset C_{\xi\delta}^i(\rho_i\beta, s).$$

where $\xi = \max\{1, \|(Df_0)^{-1}\|\}$ and $\rho_i = (\lambda_i - \rho)^{-\sigma_i^{-1}}(\lambda_i + \rho)^s$.

Proof. Let $v \in (Df_0)^{-1}(C^i_\delta(\beta, s))$. Then

$$(\lambda_i - \rho)^{\sigma_i^{-1}} \|P_i v\|^{\sigma_i^{-1}} \le \beta \left(\sum_{j \le i} (\lambda_j + \rho) \|P_j v\| \right)^s \le \beta(\lambda_i + \rho)^s \|v\|^s.$$

This implies the lemma. □

Proof of Theorem 3.10. We start by setting up notation for the Jordan form of Df_0 for $\lambda_i = 1$. It follows from our choice of the norm that there exist linear maps $Q_1, \dots, Q_{j_0}$ such that

$$\|(Df_0^k)v\| = \left\| \sum_{j=0}^{j_0} k^j Q_j v \right\| \quad \text{for } k \ge 0 \text{ and } v \in \mathrm{im}(P_i). \tag{16}$$

Let $s > 0$ be as in Lemma 3.11. Recall that $s \to \infty$ as $\nu \to 1^+$ and $\rho \to 0^+$. We choose $\rho > 0$ and $\nu > 1$ so that

$$s - \sigma_i^{-1} > 0 \quad \text{when } \lambda_i = 1,$$
$$\rho_i := (\lambda_i - \rho)^{-\sigma_i^{-1}} (\lambda_i + \rho)^s < 1 \qquad \text{when } \lambda_i < 1.$$

Let $\xi \ge 1$ be as in Lemma 3.12 and $\beta, \epsilon > 0$ as in Lemma 3.11. Take $\delta \in (0, 1)$ such that for $\|v\| < \xi\delta$, the exponential coordinates $v \mapsto \exp(v)z$, $z \in X$, are one-to-one, and

$$\|Q_j P_i v\| < \beta^{-(s - \sigma_i^{-1})^{-1}} \text{ when } \lambda_i = 1 \text{ and } j = 0, \dots, j_0. \tag{17}$$

In addition, we assume that ϵ is sufficiently small so that

$$B_\epsilon(x) \subset \exp(\{\|v\| < \delta\})x \quad \text{for all } x \in X.$$

Then by Lemma 3.11,

$$\mathcal{L}_\epsilon(x) \subset \exp(C(\beta, s) \cap \{\|v\| < \delta\})x \quad \text{for every } x \in X. \tag{18}$$

In particular,

$$\mathcal{L}_\epsilon(x) \subset \exp(C^d_\delta(\beta, s))x. \tag{19}$$

If $\lambda_d \le 1$, we argue as in the following paragraph. Otherwise, we observe that since $\delta < 1$, we have

$$C^d_\delta(\beta, s_1) \subset C^d_\delta(\beta, s_2) \quad \text{for } s_1 > s_2,$$

and hence inclusion (18) also holds for $s > 0$ such that $\rho_d = (\lambda_d - \rho)^{-\sigma_d^{-1}}(\lambda_d + \rho)^s < 1$. Applying f_0^{-1} to (19), we deduce from Lemma 3.12 that

$$\mathcal{L}_\epsilon(x) \subset \exp(C^d_{\xi\delta}(\rho_d \beta, s))x \tag{20}$$

for every $x \in X$. Using that the exponential coordinates are one-to-one, we obtain from (20) and (18) that

$$\mathcal{L}_\epsilon(x) \subset \exp(C^d_\delta(\rho_d\beta, s))x.$$

Repeating this argument, we conclude that

$$\mathcal{L}_\epsilon(x) \subset \bigcap_{k\ge 1} \exp(C^d_\delta(\rho_d^k\beta, s))x = \exp(C^d_\delta(0, s))x.$$

Now (18) implies that

$$\mathcal{L}_\epsilon(x) \subset \exp(C^{d-1}_\delta(\beta, s))x.$$

Applying the same reasoning inductively on i, we deduce that

$$\mathcal{L}_\epsilon(x) \subset \exp(C^i_\delta(0, s))x$$

provided that $\lambda_i > 1$. It follows from (18) that $\mathcal{L}_\epsilon(x) \subset \exp(C^{i-1}_\delta(\beta, s))x$.

Suppose $\lambda_i = 1$ and $\mathcal{L}_\epsilon(x) \subset \exp(C^i_\delta(\beta, s))x$ for some $\beta > 0$. We will show that

$$\mathcal{L}_\epsilon(x) \subset \exp(C^i_\delta(0, s))x.$$

Applying f_0^{-1}, we deduce that for $y = \exp(v)x \in \mathcal{L}_\epsilon(x)$, $\|v\| < \delta$, and $k \ge 0$, we have

$$\|(Df_0^k)P_iv\|^{\sigma_i^{-1}} \le \beta \left(\sum_{j<i} (\lambda_j + \rho)^k \|P_jv\| + \|(Df_0^k)P_iv\| \right)^s.$$

Using that $\lambda_j + \rho < 1$ for $j < i$ and taking $k \to \infty$, we deduce from (16) that

$$\|Q_{j_0}P_iv\|^{\sigma_i^{-1}} \le \beta \|Q_{j_0}P_iv\|^s.$$

By the choice of δ (see (17)), $\|Q_{j_0}P_iv\| = 0$. Similar arguments imply that $\|Q_jP_iv\| = 0$ for all $j = 0, \ldots, j_0$. Hence, $P_iv = 0$ and $\mathcal{L}_\epsilon(x) \subset \exp(C^i_\delta(0, s))x$. Combining this estimate with (18), we deduce that $\mathcal{L}_\epsilon(x) \subset \exp(C^{i-1}_\delta(\beta, s))x$.

Now we consider the case when $\mathcal{L}_\epsilon(x) \subset \exp(C^i_\delta(\beta, s))$ for some i such that $\lambda_i < 1$ and $\beta > 0$. Applying f_0^{-1}, it follows from Lemma 3.12 that

$$\mathcal{L}_\epsilon(x) \subset \exp(C^i_{\xi\delta}(\rho_i\beta, s))x \quad \text{for every } x \in X.$$

Then it follows from (18) that

$$\mathcal{L}_\epsilon(x) \subset \exp(C^i_\delta(\rho_i\beta, s))x,$$

and repeating this argument, we deduce that

$$\mathcal{L}_\epsilon(x) \subset \bigcap_{k\ge 1} \exp(C^i_\delta(\rho_i^k\beta, s))x = \exp(C^i_\delta(0, s))x.$$

Since the above argument can be applied inductively on i, and we conclude that $\mathcal{L}_\epsilon(x) \subset \exp(C^2_\delta(0,s))x$. This completes the proof. $\square$

Proof of Theorem 3.5. The first claim of Theorem 3.5 follows from Theorem 3.10 with $\mathcal{L}_\epsilon(x) = \mathcal{S}_{\epsilon'}(x)$ where $\mathcal{S}_{\epsilon'}(x)$ is as in (6) with sufficiently small $\epsilon' > 0$. Note that $\alpha_0(\Gamma)$ contains an essential subset by Lemma 3.9, and (11) follows from (9) where the parameter ν is close to one if f and f_0 are C^1-close.

It remains to show that Φ is bi-Hölder with respect to the metrics d^{fs}. There exists $\epsilon > 0$ such that for every $x \in X$, any points $z, w \in X$ lying on the same local leaf of W^{fs}_f in $B(x,\epsilon)$ satisfy (4). Let $\delta > 0$ be such that $\Phi(B_\delta(y)) \subset B_\epsilon(\Phi(y))$ for every $y \in X$. Consider points $z_0, w_0 \in X$ lying on the same leaf of $W^{fs}_{f_0}$ such that $d^{fs}(z_0, w_0) < \delta$. Let ℓ be a curve from z_0 to w_0 contained in $W^{fs}_{f_0}(z_0)$ such that $L(\ell) = d^{fs}(z_0, w_0)$. Then $\Phi(\ell)$ is contained in $B_\epsilon(\Phi(z_0)) \cap W^{fs}_f(\Phi(z_0))$. Moreover, since $\Phi(\ell)$ is connected, $\Phi(\ell)$ is contained in a single local leaf of W^{fs}_f in $B_\epsilon(\Phi(z_0))$. Hence,

$$d^{fs}(\Phi(z_0), \Phi(w_0)) \ll d(\Phi(z_0), \Phi(w_0)).$$

Since Φ is Hölder with respect to d, this implies that Φ is Hölder with respect to d^{fs} as well. The proof that Φ^{-1} is Hölder with respect to d^{fs} is similar. $\square$

3.3. **Convergence of the sequences $f^{-n}gf^n$.**

In this section, we study convergence of the sequence of maps $f^{-n}gf^n$ as $n \to \infty$.

First, we consider the algebraic setting:

Proposition 3.13. *Let $f_0, g_0 \in \mathrm{Aff}(X)$ be such that $Df_0 : W^{min}_{f_0} \to W^{min}_{f_0}$ is semisimple. Then*

(1) *Given a sequence $\{m_i\}$ such that*

$$(f_0^{-m_i} g_0 f_0^{m_i})(x) \to y \quad \text{as } i \to \infty$$

for some $x, y \in X$, the sequence of maps $f_0^{-m_i} g_0 f_0^{m_i} : W^{fs}_{f_0}(x) \to X$ is precompact in the C^0-topology.

(2) *There exist a sequence $\{n_i\}$ and a linear map $A : W^{min}_{f_0} \to W^{min}_{f_0}$ such that if for some $x, y \in X$ and a subsequence $\{n_{i_j}\}$,*

$$(f_0^{-n_{i_j}} g_0 f_0^{n_{i_j}})(x) \to y \quad \text{as } j \to \infty,$$

then uniformly on $v \in W^{min}_f$ in compact sets,

$$(f_0^{-n_{i_j}} g_0 f_0^{n_{i_j}}) \exp(v)x \to \exp(Av)y \quad \text{as } j \to \infty.$$

The map A is nondegenerate provided that $P^{min}_{f_0} Dg_0 : W^{min}_{f_0} \to W^{min}_{f_0}$ is nondegenerate.

If $\dim W^{min}_{f_0} = 1$, one can take $n_i = i$ and $A = P^{min}_{f_0} Dg_0$.

Proof. We have

$$(f_0^{-n} g_0 f_0^n) \exp(v) x = \exp(D(f_0^{-n} g_0 f_0^n) v)(f_0^{-n} g_0 f_0^n) x.$$

It follows from the assumption on f_0 that

$$Df_0|_{W_{f_0}^{min}} = \lambda \cdot \omega$$

where $\lambda > 0$ and ω is an isometry of $W_{f_0}^{min}$. Then

$$D(f_0^{-n} g_0 f_0^n) v = (\omega^{-n} P_{f_0}^{min}(Dg_0)\omega^n) v + (Df_0)^{-n} P_{f_0}^{max}(Dg_0)\lambda^n \omega^n v$$

where $P_{f_0}^{min}$ denotes the projection on $W_{f_0}^{min}$ and $P_{f_0}^{max}$ denotes the projection on the sum of eigenspaces complimentary to $W_{f_0}^{min}$. Since ω is an isometry, and

$$(Df_0)^{-n} P_{f_0}^{max}(Dg_0)\lambda^n \omega^n v \to 0,$$

it is clear that the sequence of maps $v \mapsto D(f_0^{-n} g_0 f_0^n) v$ is precompact in C^0-topology. This implies that the sequence $f_0^{-m_i} g_0 f_0^{m_i}|_{W_{f_0}^{fs}(x)}$ is precompact in C^0-topology as well.

To prove (2), it suffices to choose the sequence $\{n_i\}$ so that $\{\omega^{n_i}\}$ converges. This proves the proposition. □

We show that the convergence of $f_0^{-n} g_0 f_0^n|_{W_{f_0}^{fs}(x)}$ persists under small perturbations:

Theorem 3.14. *Let $f_0, g_0 \in \mathrm{Aff}(X)$ satisfy*

(i) *The map f_0 is partially hyperbolic,*
(ii) *The map $Df_0 : W_{f_0}^{min} \to W_{f_0}^{min}$ is semisimple.*

Let $f, g \in \mathrm{Diff}(X)$ be C^1-small perturbations of f_0 and g_0 and $\Phi : X \to X$ a Hölder isomorphism such that

$$\Phi \circ f_0 = f \circ \Phi \quad \textit{and} \quad \Phi \circ g_0 = g \circ \Phi$$

and

$$\Phi(W_{f_0}^{fs}(x)) = W_f^{fs}(\Phi(x)) \quad \textit{for every } x \in X.$$

Then for every $x \in X$ and a sequence $\{m_i\}$ as in Proposition 3.13(1), the sequence of maps

$$f^{-m_i} g f^{m_i} : W_f^{fs}(x) \to X, \quad i \geq 0,$$

is precompact in the C^1-topology.

Throughout this section, we assume that X is a submanifold of $\mathbb{R}^N$, which allows us to identify tangent spaces at different points.

We have a Hölder continuous decomposition (cf. (2))

$$T_x X = E_x^- \oplus E_x^+, \quad x \in X, \tag{21}$$

where $E_x^- = T_x W_f^{fs}(x)$. Let

$$P_x : T_x X \to E_x^- \quad \text{and} \quad P_x^+ : T_x X \to E_x^+$$

denote the corresponding projections.

The following proposition is the main ingredient of the proof of Theorem 3.14.

Proposition 3.15. *Let $r > 0$. Then under the assumptions of Theorem 3.14, for every $x, y \in X$ satisfying $y \in W_f^{fs}(x)$ and $d^{fs}(x, y) \le r$,*

$$\|D(f^{-n}gf^n)_x P_x - D(f^{-n}gf^n)_y P_y\| \ll d^{fs}(x, y)^\kappa \|D(f^{-n}gf^n)_x P_x\| + \delta_n$$

where $\kappa > 0$ and $\delta_n \to 0$.

Proof. Note that Φ and Φ^{-1} are also Hölder with respect to the metrics d^{fs} on the fast stable leaves of f_0 and f (see proof of Theorem 3.5). By Proposition 3.6,

$$\begin{aligned} d(f_0^{-k} g_0 f_0^n(\Phi^{-1}(x)), f_0^{-k} g_0 f_0^n(\Phi^{-1}(y))) &\ll \lambda_0^{n-k} d^{fs}(\Phi^{-1}(x), \Phi^{-1}(y)) \\ &\ll \lambda_0^{n-k} d^{fs}(x, y)^{\omega_0} \end{aligned}$$

where $\omega_0 > 0$ is the Hölder exponent of Φ^{-1} with respect to d^{fs}. Then it follows that we have the estimate

$$d(f^{-k} g f^n(x), f^{-k} g f^n(y)) \ll \lambda_0^{\omega(n-k)} d^{fs}(x, y)^{\omega_0 \omega} \tag{22}$$

where $\omega > 0$ is the Hölder exponent of Φ with respect to d.

Since the decomposition (21) is f-invariant, we have

$$P_{f(x)} D(f)_x P_x = D(f)_x P_x \quad \text{and} \quad P_{f^{-1}(x)} D(f^{-1})_x P_x = D(f^{-1})_x P_x.$$

By (3), there exist $\lambda \in (0, 1)$ and $\mu > \lambda$ such that

$$\|D(f^n)_x P_x\| \ll \lambda^n \quad \text{and} \quad \|D(f^{-n})_x P_x^+\| \ll \mu^{-n} \tag{23}$$

uniformly on $x \in X$ and $n \ge 0$. It is crucial for the proof that the map $D(f)_x P_x$ is approximately conformal (cf. assumption (ii) on f_0). Namely, for some small $\epsilon > 0$,

$$\|D(f^{-n})_x P_x\| \ll (\lambda - \epsilon)^{-n} \tag{24}$$

uniformly on $x \in X$ and $n \ge 0$. We also recall for $\rho > \lambda$ and $x, y \in X$ such that $y \in W_f^{fs}(x)$,

$$d^{fs}(f^n(x), f^n(y)) \ll \rho^n d^{fs}(x, y). \tag{25}$$

Note that the parameter ϵ in (24) satisfies $\epsilon \to 0$ as $d_{C^1}(f_0, f) \to 0$. We assume f is sufficiently close to f_0 so that

$$\zeta := (\lambda - \epsilon)^{-1} \lambda \rho^\theta < 1 \quad \text{and} \quad \nu := (\lambda - \epsilon)^{-1} \lambda \lambda_0^\omega < 1.$$

where θ is the Hölder exponent of the map $x \mapsto P_x$.

We have

$$D(f^{-n}gf^n)_x P_x = D(f^{-n})_{gf^n(x)} P_{gf^n(x)} D(g)_{f^n(x)} D(f^n)_x P_x + D(f^{-n})_{gf^n(x)} P^+_{gf^n(x)} D(g)_{f^n(x)} D(f^n)_x P_x.$$

It follows from (23) that

$$\|D(f^{-n})_{gf^n(x)} P^+_{gf^n(x)} D(g)_{f^n(x)} D(f^n)_x P_x\| \ll \lambda^n \mu^{-n} \to 0.$$

Hence, to prove the theorem, it suffices to show that for

$$A_n(x) := \left(\prod_{i=n-1}^{0} D(f^{-1})_{f^{-i}gf^n(x)}\right) P_{gf^n(x)} D(g)_{f^n(x)} \left(\prod_{i=n-1}^{0} D(f)_{f^i(x)}\right) P_x,$$

we have

$$\|A_n(x) - A_n(y)\| \ll d^{fs}(x,y)^\kappa \|A_n(x)\|.$$

We consider the operators

$$A_{n,k}(x,y) := \left(\prod_{i=n-1}^{0} D(f^{-1})_{f^{-i}gf^n(x)}\right) P_{gf^n(x)} D(g)_{f^n(x)} \times \left(\prod_{i=n-1}^{k+1} D(f)_{f^i(x)}\right) P_{f^{k+1}(x)} \left(\prod_{i=k}^{0} D(f)_{f^i(y)}\right) P_y.$$

Note that

$$\|A_n(x) - A_{n,-1}(x,y)\| \le \|A_n(x)\| \cdot \|P_x - P_x P_y\| \ll \|A_n(x)\| d(x,y)^\theta. \tag{26}$$

Now we estimate $\|A_{n,n-1}(x,y) - A_{n,-1}(x,y)\|$. We use that

$$A_{n,k}(x,y) - A_{n,k-1}(x,y) = A_n(x) B_{n,k}(x,y)$$

where

$$B_{n,k}(x,y) := \left(\prod_{i=0}^{k} D(f)^{-1}_{f^i(x)}\right) P_{f^{k+1}(x)} \left(D(f)_{f^k(y)} P_{f^k(y)} - D(f)_{f^k(x)} P_{f^k(x)}\right) \times \left(\prod_{i=k-1}^{0} D(f)_{f^i(y)}\right) P_y.$$

By (25), we have

$$\|D(f)_{f^k(y)} P_{f^k(y)} - D(f)_{f^k(x)} P_{f^k(x)}\| \ll d(f^k(x), f^k(y))^\theta \ll \rho^{\theta k} d^{fs}(x,y)^\theta,$$

and by (23) and (24),

$$\left\| \left(\prod_{i=k-1}^{0} D(f)_{f^i(y)} \right) P_y \right\| \ll \lambda^k,$$

$$\left\| \left(\prod_{i=0}^{k} D(f)^{-1}_{f^i(x)} \right) P_{f^{k+1}(x)} \right\| \ll (\lambda - \epsilon)^{-k-1}.$$

Hence,

$$\|B_{n,k}(x,y)\| \ll \zeta^k d^{fs}(x,y)^\theta$$

Since $\zeta < 1$, it follows that

$$\begin{aligned} \|A_{n,n-1}(x,y) - A_{n,-1}(x,y)\| &\leq \sum_{k=0}^{n-1} \|A_{n,k}(x,y) - A_{n,k-1}(x,y)\| \\ &\ll \|A_n(x)\| d^{fs}(x,y)^\theta. \end{aligned} \tag{27}$$

We claim that for some $c > 0$ and all $k = -1, \ldots, n-1$,

$$\|A_{n,k}(x,y)\| \ll (1 + d^{fs}(x,y)^\theta) \cdot \|A_n(x)\|. \tag{28}$$

Setting

$$C_k(x,y) := \left(\prod_{i=0}^{k} D(f)^{-1}_{f^i(x)} \right) P_{f^{k+1}(x)} \left(\prod_{i=k}^{0} D(f)_{f^i(y)} \right) P_y,$$

we have

$$A_{n,k}(x,y) = A_n(x) C_k(x,y).$$

Now equation (28) will follow from the estimate

$$\|C_k(x,y)\| \ll 1 + d^{fs}(x,y)^\theta.$$

In fact, we will show that

$$\|C_k(x,y) - P_x P_y\| \ll d^{fs}(x,y)^\theta. \tag{29}$$

Using (23) and (24), we deduce that

$$\begin{aligned} &\|C_k(x,y) - C_{k-1}(x,y)\| \\ =& \left\| \left(\prod_{i=0}^{k-1} D(f)^{-1}_{f^i(x)} \right) P_{f^k(x)} \left(D(f)^{-1}_{f^k(x)} D(f)_{f^k(y)} - id \right) \right. \\ & \qquad \left. \times \left(\prod_{i=k-1}^{0} D(f)_{f^i(y)} \right) P_y \right\| \\ \ll& (\lambda - \epsilon)^{-k} d(f^k(x), f^k(y))^\theta \lambda^k \ll \zeta^k d^{fs}(x,y)^\theta. \end{aligned}$$

Since $C_{-1}(x,y) = P_x P_y$ and $\zeta < 1$, the last estimate implies (29) and (28).

Next, we consider the operators

$$D_{n,k}(x,y) := \left(\prod_{i=n-1}^{k} D(f^{-1})_{f^{-i}gf^n(y)}\right) P_{f^{-k}gf^n(y)} \left(\prod_{i=k-1}^{0} D(f^{-1})_{f^{-i}gf^n(x)}\right)$$
$$\times P_{gf^n(x)} D(g)_{f^n(x)} P_{f^n(x)} \left(\prod_{i=n-1}^{0} D(f)_{f^i(y)}\right) P_y.$$

Using (22), we deduce that

$$\begin{aligned}\|A_{n,n-1}(x,y) - D_{n,n}(x,y)\| &\le \|P_{f^{-n}gf^n(x)} - P_{f^{-n}gf^n(y)}P_{f^{-n}gf^n(x)}\| \cdot \|A_{n,n-1}(x,y)\| \\ &\ll d(f^{-n}gf^n(x), f^{-n}gf^n(y))^\theta \|A_{n,n-1}(x,y)\| \\ &\ll d^{fs}(x,y)^{\theta\omega_0\omega} \|A_{n,n-1}(x,y)\| \\ &\ll d^{fs}(x,y)^{\theta\omega_0\omega} \|A_n(x)\|.\end{aligned}$$

To estimate $\|D_{n,n}(x,y) - D_{n,0}(x,y)\|$, we use the argument similar to the proof of (27). We have

$$D_{n,k}(x,y) - D_{n,k-1}(x,y) = E_{n,k}(x,y) A_{n,n-1}(x,y)$$

where

$$\begin{aligned}E_{n,k}(x,y) := &\left(\prod_{i=n-1}^{k} D(f^{-1})_{f^{-i}gf^n(y)}\right) P_{f^{-k}gf^n(y)} \\ &\times \left(D(f^{-1})_{f^{-(k-1)}gf^n(x)} P_{f^{-(k-1)}gf^n(x)} - D(f^{-1})_{f^{-(k-1)}gf^n(y)} P_{f^{-(k-1)}gf^n(y)}\right) \\ &\times \left(\prod_{i=k-1}^{n-1} D(f^{-1})^{-1}_{f^{-i}gf^n(x)}\right) P_{f^{-n}gf^n(x)}\end{aligned}$$

Applying (24), (22), and (23), we deduce that

$$\|E_{n,k}(x,y)\| \ll \nu^{n-k} d^{fs}(x,y)^{\theta\omega_0\omega}.$$

Since $\nu < 1$, it follows that

(30)
$$\|D_{n,n}(x,y) - D_{n,0}(x,y)\| \le \sum_{k=1}^{n} \|D_{n,k}(x,y) - D_{n,k-1}(x,y)\|$$

(31)
$$\ll d^{fs}(x,y)^{\theta\omega_0\omega} \|A_{n,n-1}(x,y)\| \ll d^{fs}(x,y)^{\theta\omega_0\omega} \|A_n(x)\|.$$

Next, we compare the maps $A_n(y)$ and $D_{n,0}(x,y)$:

$$\|A_n(y) - D_{n,0}(x,y)\| = \Bigg\| \left(\prod_{i=n-1}^{0} D(f^{-1})_{f^{-i}gf^n(y)} \right) P_{gf^n(y)} \times \left(P_{gf^n(y)} D(g)_{f^n(y)} P_{f^n(y)} - P_{gf^n(x)} D(g)_{f^n(x)} P_{f^n(x)}\right) \times \left(\prod_{i=n-1}^{0} D(f)_{f^i(y)} \right) P_y \Bigg\|.$$

We have

$$\left\|P_{gf^n(y)} D(g)_{f^n(y)} P_{f^n(y)} - P_{gf^n(x)} D(g)_{f^n(x)} P_{f^n(x)}\right\| \ll d(f^n(x), f^n(y))^\theta \ll \rho^{\theta n} d^{fs}(x,y)^\theta.$$

Combining this estimate with (23) and (24), we deduce that

$$\|A_n(y) - D_{n,0}(x,y)\| \ll \zeta^n d^{fs}(x,y)^\theta.$$

Finally, the proposition follows from the estimate

$$\begin{aligned}\|A_n(x) - A_n(y)\| \le & \|A_n(x) - A_{n,-1}(x,y)\| + \|A_{n,-1}(x,y) - A_{n,n-1}(x,y)\| \\ & + \|A_{n,n-1}(x,y) - D_{n,n}(x,y)\| + \|D_{n,n}(x,y) - D_{n,0}(x,y)\| \\ & + \|D_{n,0}(x,y) - A_n(y)\|.\end{aligned}$$

This completes the proof. □

Proposition 3.16. *Let $x_0 \in X$ and $r > 0$. Then under the assumptions of Theorem 3.14,*

$$\sup\{\|D(f^{-n}gf^n)_x P_x\| : \ x \in W_f^{fs}(x_0),\ d^{fs}(x,x_0) \le r,\ n \in \mathbb{N}\} < \infty.$$

Proof. Suppose that the claim fails, i.e., there exist sequences $x_i \in W_f^{fs}(x_0)$, $d^{fs}(x_i,x_0) \le r$, and $n_i \in \mathbb{N}$, $n_i \to \infty$, such that

$$\|D(f^{-n_i}gf^{n_i})_{x_i} P_{x_i}\| \to \infty.$$

Passing to a subsequence, we may assume that $x_i \to x_\infty$ for some $x_\infty \in W_f^{fs}(x_0)$ such that $d^{fs}(x_\infty, x_0) \le r$. It follows from Proposition 3.15 that

$$\|D(f^{-n_i}gf^{n_i})_{x_\infty} P_{x_\infty}\| \ge (1 - c \cdot d^{fs}(x_i, x_\infty)^\kappa) \|D(f^{-n_i}gf^{n_i})_{x_i} P_{x_i}\| - \delta_{n_i} \to \infty.$$

Let $v_i \in T_{x_\infty}(W_f^{fs}(x_\infty))$ with $\|v_i\| = 1$ be such that

$$\|D(f^{-n_i}gf^{n_i})_{x_\infty} P_{x_\infty}\| = \|D(f^{-n_i}gf^{n_i})_{x_\infty} v_i\|.$$

Passing to a subsequence, we may assume that $v_i \to v_\infty$. We have

$$\begin{aligned}\|D(f^{-n_i}gf^{n_i})_{x_\infty}v_\infty\| &\geq \|D(f^{-n_i}gf^{n_i})_{x_\infty}v_i\| - \|D(f^{-n_i}gf^{n_i})_{x_\infty}P_{x_\infty}(v_\infty - v_i)\| \\ &\geq \|D(f^{-n_i}gf^{n_i})_{x_\infty}P_{x_\infty}\| \cdot (1 - \|v_\infty - v_i\|).\end{aligned}$$

Hence, for sufficiently large i, we have

$$\|D(f^{-n_i}gf^{n_i})_{x_\infty}v_\infty\| \geq \frac{1}{2}\|D(f^{-n_i}gf^{n_i})_{x_\infty}P_{x_\infty}\|.$$

Let $\alpha_n = \|D(f^{-n}gf^{n})_{x_\infty}v_\infty\|$. Note that $\alpha_{n_i} \to \infty$.

Fix small $\epsilon > 0$. Let $x \in W_f^{fs}(x_\infty)$ be such that $d^{fs}(x, x_\infty) < \epsilon$ and $v \in T_xW_f^{fs}(x)$ such that $\|v - v_\infty\| < \epsilon$. We have

$$\begin{aligned}\|D(f^{-n}gf^{n})_x v - D(f^{-n}gf^{n})_{x_\infty}v_\infty\| \leq &\|D(f^{-n}gf^{n})_x P_x v - D(f^{-n}gf^{n})_{x_\infty}P_{x_\infty}v\| \\ &+ \|D(f^{-n}gf^{n})_{x_\infty}P_{x_\infty}v - D(f^{-n}gf^{n})_{x_\infty}P_{x_\infty}v_\infty\| \\ \ll &d^{fs}(x, x_\infty)^\kappa\|D(f^{-n}gf^{n})_{x_\infty}P_{x_\infty}\| + \delta_n \\ &+ \|D(f^{-n}gf^{n})_{x_\infty}P_{x_\infty}\| \cdot \|v - v_\infty\| \\ \ll &(\epsilon^\kappa\alpha_n + \delta_n) + \epsilon\alpha_n.\end{aligned}$$

Let $\ell : [0, 1] \to W_f^{fs}(x_\infty)$ be a smooth curve such that

$$\begin{aligned}&\ell(0) = x_\infty, \quad \ell'(0) = v_\infty, \quad \ell'(t) \in T_{\ell(t)}W_f^{fs}(\ell(t)), \\ &\mathrm{diam}(\ell([0, 1])) < \epsilon, \qquad \|\ell'(t) - \ell'(0)\| < \epsilon.\end{aligned}$$

We consider the sequence of curves $\ell_n = (f^{-n}gf^{n})\ell$. Note that $\|\ell_n'(0)\| = \alpha_n$, and it follows from the previous computation that, choosing ϵ sufficiently small,

$$\|\ell_{n_i}'(t) - \ell_{n_i}'(0)\| \leq \frac{1}{3}\|\ell_{n_i}'(0)\|$$

for sufficiently large i. Since $\|\ell_{n_i}'(0)\| \to \infty$, it follows that the distance between $\ell_{n_i}(0)$ and $\ell_{n_i}(1)$ in the ambient Eucledean space goes to infinity as $i \to \infty$. This contradiction proves the proposition. □

Proof of Theorem 3.14. By Proposition 3.13(1), the maps $\{f^{-m_i}gf^{m_i}|_{W_f^{fs}(x)}\}$ are precompact in C^0-topology. Then it follows from Proposition 3.16 that the maps $\{f^{-m_i}gf^{m_i}|_{W_f^{fs}(x)}\}$ are uniformly bounded in the C^1-topology. Also, combining Proposition 3.15 and Proposition 3.16, we obtain that for every z and w in a compact neighborhood of x in $W_f^{fs}(x)$,

$$\|D(f^{-m_i}gf^{m_i})_z P_z - D(f^{-m_i}gf^{m_i})_w P_w\| \ll d^{fs}(z, w)^\kappa + \delta_{m_i}.$$

Since $\delta_{m_i} \to 0$, it follows that the maps $\{f^{-m_i}gf^{m_i}|_{W_f^{fs}(x)}\}$ are equicontinuous in the C^1-topology. This implies the theorem. □

3.4. **Hölder implies C^∞ along fast stable manifolds.**

Theorem 3.17. *Let $f_0, g_0 \in \mathrm{Aff}(X)$ be a good pair, $f, g \in \mathrm{Diff}(X)$ C^1-small perturbations of f_0, g_0, and $\Phi : X \to X$ a Hölder isomorphism such that*

$$\Phi \circ f_0 = f \circ \Phi \quad \textit{and} \quad \Phi \circ g_0 = g \circ \Phi,$$

and

$$\Phi(W^{fs}_{f_0}(x)) = W^{fs}_f(\Phi(x)) \quad \textit{for all } x \in X.$$

Then for a.e. $x \in X$, the maps $\Phi|_{W^{fs}_{f_0}(x)}$ and $\Phi^{-1}|_{W^{fs}_f(\Phi(x))}$ are C^∞-diffeomorphisms.

Proof. Fix a sequence $\{n_i\}$ and $A \in \mathrm{GL}(W^{min}_{f_0})$ as in Proposition 3.13(2). For a set of $x \in X$ of full measure, the sequence $\{f_0^{-n_i} g_0 f_0^{n_i}(x)\}$ is dense in X. In particular, for a.e. $x \in X$ and every $y \in W^{fs}_f(x)$, there exists a subsequence $\{n_{i_j}\}$ such that $f_0^{-n_{i_j}} g_0 f_0^{n_{i_j}}(x) \to y$. Then by Proposition 3.13(2), for every $z = \exp(v)x$ with $v \in W^{min}_{f_0}$,

$$f_0^{-n_{i_j}} g_0 f_0^{n_{i_j}}(z) \to \exp(Av)y \tag{32}$$

uniformly on compact sets.

For $k \in \mathbb{N}$ and $y \in W^{fs}_{f_0}(x)$, we consider maps

$$\rho^0_{k,y} : W^{fs}_{f_0}(x) \to W^{fs}_{f_0}(x) : \exp(v)x \mapsto \exp(A^k v)y,$$
$$\rho^1_{k,y} : W^{fs}_f(\Phi(x)) \to W^{fs}_f(\Phi(x)) : \Phi(\exp(v)x) \mapsto \Phi\left(\exp(A^k v)y\right),$$

where $v \in W^{min}_{f_0}$. Note that

$$\rho^0_{k,y} = \Phi^{-1} \circ \rho^1_{k,y} \circ \Phi. \tag{33}$$

In particular, it follows that $\rho^1_{k,y}$ is a homeomorphism, and by (32),

$$\rho^1_{1,y} = \lim_{j\to\infty} (f^{-n_{i_j}} g f^{n_{i_j}})|_{W^{fs}_f(\Phi(x))}.$$

in the C^0-topology. By Theorem 3.14, there exists a subsequence which converges in the C^1-topology. Hence, $\rho^1_{1,y}$ is a C^1-map for every $y \in W^{fs}_{f_0}(x)$. Since each map $\rho^1_{k,y}$, $k \geq 1$, is a composition of maps $\rho^1_{1,z}$, it is also C^1.

Next, we show that

$$D(\rho^1_{1,y})_z \neq 0 \quad \text{for every } y \in W^{fs}_{f_0}(x) \text{ and } z \in W^{fs}_f(\Phi(x)). \tag{34}$$

Suppose that, to the contrary, $D(\rho^1_{1,y_0})_{z_0} = 0$ for some $y_0 \in W^{fs}_{f_0}(x)$ and $z_0 \in W^{fs}_f(\Phi(x))$. For every $y \in W^{fs}_{f_0}(x)$, there exists $y_1 \in W^{fs}_{f_0}(x)$ such that

$$\rho^0_{2,y} = \rho^0_{1,y_1}\rho^0_{1,y_0},$$

and by (33),

$$\rho^1_{2,y} = \rho^1_{1,y_1}\rho^1_{1,y_0}.$$

Hence, $D(\rho^1_{2,y})_{z_0} = 0$ for every $y \in W^{fs}_{f_0}(x)$. Similarly, using (33), we deduce that for every $z \in W^{fs}_f(\Phi(x))$, there exists $y_z \in W^{fs}_{f_0}(x)$ such that $\rho^1_{1,y_z}(z) = z_0$. If we fix $y_2 \in W^{fs}_{f_0}(x)$, there exists $y'_z \in W^{fs}_{f_0}(x)$ such that

$$\rho^1_{3,y_2} = \rho^1_{2,y'_z}\rho^1_{1,y_z}.$$

Then we have

$$D(\rho^1_{3,y_2})_z = 0 \quad \text{for every } z \in W^{fs}_f(\Phi(x)).$$

This contradicts the map ρ^1_{3,y_2} being a homeomorphism, and (34) follows. We have proved that $\rho^1_{1,y}$ is a C^1-diffeomorphism for every $y \in W^{fs}_{f_0}(x)$. This implies that the map $\rho^1_{0,y}$, which can be represented as a composition of ρ^1_{1,z_1} and $(\rho^1_{1,z_2})^{-1}$, is also a C^1-diffeomorphism for every $y \in W^{fs}_{f_0}(x)$.

We have a free transitive C^0-action of $W^{min}_{f_0}$ on $W^{fs}_f(\Phi(x))$ defined by

$$s(v, \Phi(\exp(w)x)) = \Phi(\exp(v+w)x) = \rho^1_{0,\exp(v)x}(\Phi(\exp(w)x)) \tag{35}$$

where $v, w \in W^{min}_{f_0}$. For every $v \in W^{min}_{f_0}$, the map $s(v,\cdot)$ is a C^1-diffeomorphism. It follows from the Bochner–Montgomery theorem [3] that the map

$$s : W^{min}_{f_0} \times W^{fs}_f(\Phi(x)) \to W^{fs}_f(\Phi(x))$$

is C^1 as well. This implies that $\Phi : W^{fs}_{f_0}(x) \to W^{fs}_f(\Phi(x))$ and its inverse are C^1-maps.

For the next step, we first assume that $\dim W^{fs}_{f_0}(x) = 1$. For $y \in W^{fs}_{f_0}(x)$, we define the map $\alpha^0_{x,y} : W^{fs}_{f_0}(x) \to W^{fs}_{f_0}(x)$ by

$$\alpha^0_{x,y} : \exp(w)x \mapsto \exp(Aw)y$$

where $w \in W^{min}_{f_0}$ and $A = P^{min}_{f_0} Dg_0$. Also, we set

$$\alpha^1_{x,y} = \Phi \circ \alpha^0_{x,y} \circ \Phi^{-1} : W^{fs}_f(\Phi(x)) \to W^{fs}_f(\Phi(x)).$$

Our aim is to show that the map $\alpha^1_{x,y}$ is a C^∞-diffeomorphism for a.e. $x \in X$ and every $y \in W^{fs}_{f_0}(x)$. Consider the measurable function

$$\sigma(x) = \sup\{\|D(f^{-n}gf^n)_{\Phi(x)}P_{\Phi(x)}\| : \ n \in \mathbb{N}\},$$

which is well defined by Proposition 3.16. For $c > 0$, let $X(c)$ be the subset of $x \in X$ such that $\sigma(x) \le c$ and the sequence $\{f_0^{-n}g_0f_0^n(x)\}$ is dense in X. By property (iv) of good pair and Proposition 3.16, the set $\cup_{c>0}X(c)$ has full measure in X. Hence, it is suffices to show that $\alpha^1_{x,y}$ is a C^∞-diffeomorphism for a.e. $x \in X(c)$ and every $y \in W^{fs}_{f_0}(x)$.

Using the Poincare recurrence theorem, for a.e. $x \in X(c)$, we construct sequences $\{k_j\}$, $k_1 = 1$, and $\{n_i^{(j)}\}$ such that

$$f_0^{k_j}(x) \in X(c) \quad \text{for every } j \geq 1,$$
$$(f_0^{-n_i^{(j)}} g_0 f_0^{n_i^{(j)}}) f_0^{k_j}(x) \to f_0^{k_j}(y) \quad \text{as } i \to \infty.$$

Then by Proposition 3.13(2),

$$(f_0^{-n_i^{(j)}} g_0 f_0^{n_i^{(j)}})|_{W^{fs}_{f_0}(f_0^{k_j}(x))} \to \alpha^0_{f_0^{k_j}(x), f_0^{k_j}(y)} \quad \text{as } i \to \infty \tag{36}$$

in the C^0-topology. Since $f_0^{k_j} \circ \alpha^0_{x,y}$ and $\alpha^0_{f_0^{k_j}(x), f_0^{k_j}(y)} \circ f_0^{k_j}$ are both affine maps, and since they have the same slope (here we used that $\dim W^{fs}_{f_0}(x) = 1$) and map x to $f_0^{k_j}(y)$, it follows that

$$f_0^{k_j} \circ \alpha^0_{x,y} = \alpha^0_{f_0^{k_j}(x), f_0^{k_j}(y)} \circ f_0^{k_j},$$

and hence,

$$f^{k_j} \circ \alpha^1_{x,y} = \alpha^1_{f_0^{k_j}(x), f_0^{k_j}(y)} \circ f^{k_j}. \tag{37}$$

By the nonstationary Sternberg linearization [13, 12], there exist a family of C^∞-diffeomorphisms

$$L_z : \mathbb{R} \to W^{fs}_f(z), \quad z \in X,$$

such that the map $z \mapsto L_z$ is continuous in the C^∞-topology, $L_z'(0) = 1$, and

$$(L^{-1}_{f(z)} \circ f \circ L_z)(t) = \rho(z) t \tag{38}$$

with $|\rho(z)| < 1$. Consider the sequence of maps

$$g_k = L^{-1}_{f^k(\Phi(x))} \circ \alpha^1_{f_0^k(x), f_0^k(y)} \circ L_{f^k(\Phi(x))} : \mathbb{R} \to \mathbb{R}.$$

We claim that the sequence of maps g_{k_j} restricted to compact sets is uniformly bounded and equicontinuous in the C^1-topology. This is equivalent to the sequence $\{\alpha^1_{f_0^{k_j}(x), f_0^{k_j}(y)}\}$ being uniformly bounded and equicontinuous. It follows from (36) that

$$F_i := (f^{-n_i^{(0)}} g f^{n_i^{(0)}})|_{W^{fs}_f(\Phi(x))} \to \alpha^1_{x,y} \quad \text{as } i \to \infty$$

in the C^0-topology, and by Theorem 3.14, we may assume, after passing to a subsequence, that convergence also holds in the C^1-topology. By (37),

$$\begin{aligned}\alpha^1_{f_0^{k_j}(x),f_0^{k_j}(y)} =&(f^{k_j}\circ\alpha^1_{x,y}\circ f^{-k_j})|_{W_f^{fs}(f^{k_j}(\Phi(x)))}\\ =&(f^{k_j}\circ(\alpha^1_{x,y}-F_i)\circ f^{-k_j})|_{W_f^{fs}(f^{k_j}(\Phi(x)))}\\ &+(f^{-(n_i^{(0)}-k_j)}gf^{n_i^{(0)}-k_j})|_{W_f^{fs}(f^{k_j}(\Phi(x)))}.\end{aligned}$$

Taking $i=i(j)$ sufficiently large, the first term can be made arbitrary small in the C^1-topology, and since $f_0^{k_j}(x)\in X(c)$ for all j, the derivative of the second term is uniformly bounded in the C^1-topology. This proves that the sequence $\{\alpha^1_{f_0^{k_j}(x),f_0^{k_j}(y)}\}$ is uniformly bounded. To prove equicontinuity, we observe that for $z,w\in W_f^{fs}(f^{k_j}(\Phi(x)))$,

$$\begin{aligned}&\left\|D(\alpha^1_{f_0^{k_j}(x),f_0^{k_j}(y)})_z-D(\alpha^1_{f_0^{k_j}(x),f_0^{k_j}(y)})_w\right\|\\ \le&\|D(f^{k_j}\circ(\alpha^1_{x,y}-F_i)\circ f^{-k_j})_zP_z\|\\ &+\|D(f^{k_j}F_if^{-k_j})_zP_z-D(f^{k_j}F_if^{-k_j})_wP_w\|\\ &+\|D(f^{k_j}\circ(F_i-\alpha^1_{x,y})\circ f^{-k_j})_wP_w\|.\end{aligned}$$

Since $F_i\to\alpha^1_{x,y}$ in the C^1-topology, taking $i=i(j)$ sufficiently large, we can make the first and the last terms arbitrary small. To estimate the middle term, we use that $f_0^{k_j}(x)\in X(c)$ for all j and Proposition 3.15. We get

$$\|D(f^{k_j}F_if^{-k_j})_zP_z-D(f^{k_j}F_if^{-k_j})_wP_w\|\ll d(z,w)^\kappa+\delta_{n_i^{(0)}},$$

where $\delta_n\to 0$. This proves equicontinuity.

Let $\rho_k=\prod_{s=0}^{k-1}\rho(f^s(\Phi(x)))$ with ρ defined as in (38). Note that $\rho_k\to 0$ uniformly on $x\in X$. We deduce from (37) and (38) that

$$g_0(t)=\rho_{k_j}^{-1}g_{k_j}(\rho_{k_j}t). \tag{39}$$

Since the sequence $\{g_{k_j}\}$ is equicontinuous in C^1-topology, for every sequence $\theta_j\to 0$, we have

$$\|g'_{k_j}(\theta_j)-g'_{k_j}(0)\|\to 0\quad\text{as } j\to\infty. \tag{40}$$

Hence, applying the mean value theorem to (39) and taking $j\to\infty$, we deduce that

$$g_0(t)=\left(\limsup_{j\to\infty}g'_{k_j}(0)\right)t$$

In particular, g_0 and $\alpha^1_{x,y}$ are a C^∞-diffeomorphism for every $y \in W^{fs}_{f_0}(x)$. Now the Bochner–Montgomery theorem [3], applied to the action s defined in (35) is C^∞. This implies that $\Phi : W^{fs}_{f_0}(x) \to W^{fs}_f(\Phi(x))$ and its inverse are C^∞-maps, which completes the proof of the theorem under the assumption that $\dim W^{fs}_{f_0}(x) = 1$.

Now we consider the case when $\dim W^{fs}_{f_0}(x) \geq 2$. For every $x \in X$, the maps $\rho^0_{0,y}$, $y \in W^{fs}_{f_0}(x)$, define a transitive conformal (in fact, affine) action s_0 on $W^{fs}_{f_0}(x)$. Also, for a.e. $x \in X$, the maps $\rho^1_{0,y}$, $y \in W^{fs}_{f_0}(x)$, define a transitive C^1-action s_1 on $W^{fs}_f(\Phi(x))$. The actions s_0 and s_1 are conjugate via Φ. Recall that we have already proved that $\Phi|_{W^{fs}_{f_0}(x)}$ and its inverse are C^1-maps for a.e. $x \in X$. Hence, the image of the standard conformal structure on $W^{fs}_{f_0}(x)$ under Φ defines a C^1-conformal structure on $W^{fs}_f(\Phi(x))$. Clearly, the action s_1 is conformal with respect to this structure. Shefel [25] proved that every conformal transformation of a C^k-conformal structure with $k \geq 1$ is C^{k+1}. Hence, each element of the s_1 action acts by C^2-maps. Using transitivity and the Bochner–Montgomery theorem [3], we deduce that s_1 is a C^2-action. Since $\Phi|_{W^{fs}_{f_0}(x)}$ conjugates s_0 and s_1, it follows that $\Phi|_{W^{fs}_{f_0}(x)}$ is a C^2-diffeomorphism. Continuing this argument we deduce that $\Phi|_{W^{fs}_{f_0}(x)}$ and its inverse are C^∞. This proves the theorem. $\square$

3.5. **Completion of the proof of the main theorem.** From the results of the previous subsections, we see that $\Phi|_{W^{fs}_f(x)}$ is a C^∞-diffeomorphism for every good $f \in \alpha_0(\Gamma)$ and a.e. $x \in X$. Note that if f is good, then $g^{-1}fg$ is good as well for every $g \in \alpha_0(\Gamma)$, and $W^{min}_{g^{-1}fg} = (Dg)^{-1}W^{min}_f$. Hence, it follows from the irreducibility of the Γ action on $\mathrm{Lie}(G)$ that

$$\sum_{f\in\alpha_0(\Gamma)\text{-good}} W^{min}_f = \mathrm{Lie}(G), \tag{41}$$

and for every $x \in X$,

$$\sum_{f\in\alpha_0(\Gamma)\text{-good}} T_x(W^{fs}_f(x)) = T_xX.$$

Note also that the fast stable foliations of f and its conjugate are absolutely continuous. Finally, to deduce that Φ is C^∞-diffeomorphism, we apply either [5, Theorem 3] or [23, Theorem 1.1].

4. Existence of good pairs

4.1. **Tori.** In this section, we set $X = \mathbb{T}^d$, $d \geq 2$, and prove

Proposition 4.1. *Let Γ be a subgroup of $\mathrm{Aff}(X)$ such that the Zariski closure of $D\Gamma$ contains SL_d. Then Γ contains a good pair.*

We will use the following lemma, which is easy to prove using Fourier analysis (see, for example, [2, Corollary 1.6 and Remark 1.8]). Let ϕ be the Euler totient function.

Lemma 4.2. *Let $f_1, f_2 \in \mathrm{Aff}(X)$ be such that for every $l \geq 1$ satisfying $\phi(l) \leq d^2$, the map $Df_1^{-l} Df_2^l$ does not have eigenvalue 1. Then for every $\phi_1, \phi_2 \in L^2(X)$,*

$$\int_X \phi_1(f_1^n(x))\phi_2(f_2^n(x))d\mu(x) \to \left(\int_X \phi_1\, d\mu\right)\left(\int_X \phi_2\, d\mu\right) \quad \text{as } n \to \infty.$$

If the conclusion of Lemma 4.2 holds, then we call the pair $\{f_1, f_2\}$ *mixing*. Mixing pairs can be used to construct affine maps satisfying property (iv) of good pairs.

Lemma 4.3. *Let $f, g \in \mathrm{Aff}(X)$ and suppose the pair $\{f^{-1}, gf^{-1}g^{-1}\}$ is mixing. Then for every subsequence $\{n_i\}$ and for a.e. $x \in X$, the sequence $\{f^{-n_i} g f^{n_i}(x)\}_{n \geq 0}$ is dense in X.*

Proof. We have

$$\int_X \phi_1(gf^{-n}g^{-1}(x))\phi_2(f^{-n}(x))d\mu(x) \to \left(\int_X \phi_1\, d\mu\right)\left(\int_X \phi_2\, d\mu\right) \quad \text{as } n \to \infty$$

for every $\phi_1, \phi_2 \in L^2(X)$. By invariance of the measure, this also implies that

$$\int_X \phi_1(x)\phi_2(f^{-n}gf^n(x))d\mu(x) \to \left(\int_X \phi_1\, d\mu\right)\left(\int_X \phi_2\, d\mu\right) \quad \text{as } n \to \infty$$

for every $\phi_1, \phi_2 \in L^2(X)$.

Now we show that for $\delta_n = f^{-n}gf^n$, the sequence $\{\delta_{n_i} x\}$ is dense in X for a.e. $x \in X$. Let U be a nonempty open subset of X and $A = \cup_{i \geq 0} \delta_{n_i}^{-1}(U)$. We have

$$0 = \int_X \chi_U(\delta_{n_i}(x))\chi_{A^c}(x)\, d\mu(x) \to \mu(U)\mu(A^c).$$

This implies that $\mu(A^c) = 0$, i.e. for a.e. $x \in X$,

$$\{\delta_{n_i} x\}_{i \geq 0} \cap U \neq \emptyset.$$

Since X has countable base of topology, this proves the lemma. $\square$

Proof of Proposition 4.1. Since $D\Gamma$ is Zariski dense, there is $f \in \Gamma$ such that Df is $\mathbb{R}$-regular (see [1, 22]). In particular, Df is semisimple and hyperbolic. Because of Lemmas 4.2 and 4.3, it suffices to find $g \in \Gamma$ such that Dg belongs to the set

$$\left\{X \in \mathrm{SL}_d : \ \det(P_f^{min} X|_{W_f^{min}}) \neq 0, \quad \det([Df^l, X] - id) \neq 0 \ \text{ for } \phi(l) \leq d^2\right\}.$$

One can check that this is a nonempty Zariski open subset of SL_d. Hence, existence of such $g \in \Gamma$ follows from Zariski density. □

4.2. **Semisimple groups.** Let G be a connected semisimple Lie groups with no compact factors, Λ a lattice in G, and $X = G/\Lambda$.

Proposition 4.4. *Let Γ be a subgroup of $\mathrm{Aff}(X)$ such that the Zariski closure of $D\Gamma$ contains $\mathrm{Ad}(G)$. Then Γ contains a good pair.*

Proof. Since $D\Gamma$ contains a finite index subgroup consisting of inner automorphisms, we may assume without loss of generality that $D\Gamma$ is a subgroup of $\mathrm{Ad}(G)$. It follows from Zariski density [1, 22] that Γ contains an element f such that Df is $\mathbb{R}$-regular. In particular, it is partially hyperbolic and semisimple, and hence it satisfies properties (i)–(ii) of the definition of a good pair. If we choose $g \in \Gamma$ so that the pair $\{f^{-1}, gf^{-1}g^{-1}\}$ is mixing, then by Lemma 4.3, f and g will satisfy property (iv) of the definition of a good pair. By the Howe–Moore theorem, the pair $\{f^{-1}, gf^{-1}g^{-1}\}$ is mixing provided that for all projections $\pi_i : \mathrm{Ad}(G) \to \mathrm{Ad}(G_i)$ on simple factors of $\mathrm{Ad}(G)$, the sequence $\{\pi_i(Dg(Df)^{-n}(Dg)^{-1}(Df)^n)\}$ is divergent. Since $\pi_i(Df)$ is also $\mathbb{R}$-regular,

$$P_i = \{g \in G_i : \ \pi_i(Df)^{-n} \cdot g \cdot \pi_i(Df)^n \text{ is nondivergent}\}$$

is a proper parabolic subgroup of G_i. By Zariski density, there exists $g \in \Gamma$ such that $\pi_i(Dg) \notin P_i$ for all i, and $P_f^{min}(Dg) : W_f^{min} \to W_f^{min}$ is nondegenerate. Such f and g provide a good pair. □

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