Homework 1 - Solutions

(1) If two balanced dice are rolled, what is the probability that the sum of spots is equal to five? Describe the sample space and the event as a subset of the sample space.
Solution: The sample space is pairs of integers between 1 and 6. There are 36 of them. 4 of these pairs add up to 6. Hence the probability is 4/36 or 1/9.

(2) In how many ways can 11 one dollar coins be divided amongst 7 children? What if each child gets at least one coin?
Solution: This corresponds to the problem of writing 11 as the sum of 7 nonnegative integers: $11 = \sum_{i=1}^{7} x_i$. By Proposition 6.2 form the book, there are $\binom{17}{6}$ many solutions. If every child gets at least one coin, there are $\binom{10}{6}$ many solutions.

(3) A drawer contains 4 pairs of socks, one red, one black, one white and one green. If 4 socks are selected at random, what is the probability that the 4 socks include at least one pair?
Solution: There are $\binom{8}{4=70}$ possibilities to pick 4 socks out of 8 (4 pairs). Let's first calculate the probability that we are not choosing a pair at all. Then we have 8 possibilities for the first choice, 6 for the next, then 4, then 2. Since order does not matter, we get $8 \cdot 6 \cdot 4 \cdot 2/4!$ many possibiliites. Hence the chances that we are picking at least one pair are $1 - \frac{16}{70}$.

(4) What is the probability that a (5 card) poker hand contains a pair (i.e. at least two cards of the same denomination). Here assume that we just pick 5 cards out of a deck at random.
Solution: The total number of 5 card poker hands is $\binom{52}{5}$. The number of possibilities of not getting at least a pair is $52 \cdot 48 \cdot 44 \cdot 40 \cdot 36$. Hence the probability of getting a pair is:

$$1 - \frac{52 \cdot 48 \cdot 44 \cdot 40 \cdot 36}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$$

(5) Denote by $A - B := A \cap B^c$. Define the symmetric difference between two events A and B in a sample space S as $A \triangle B := (A \cup B) - AB$. This is the event that exactly one of A or B occur. Let P be a probability on S. Show that for any two events A and B

$$P(A \triangle B) = P(A) + P(B) - 2P(AB).$$

Solution: Note that $A \cup B = A \triangle B \cup A \cdot B$ is a union of mutually exclusive events. Hence we get

$$P(A \cup B) = P(A \triangle B) + P(A \cdot B$$

as desired.

(6) Suppose A and B are mutually exclusive events for which $P(A) = .3$ and $P(B) = .5$. What is the probability that
(a) either A or B occurs; **Solution:** .8
(b) A occurs but B does not; **Solution:** .3
(c) both A and B occur. **Solution:** 0

(7) Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \ldots + \binom{n}{r}\binom{m}{0}$$

Hint: Consider a group of n men and m women. How many groups of size r are possible?

Solution: Divide r into the sums of i men and $r-i$ women. Then we get $\binom{n}{i}$ possibilities for picking i men and $\binom{m}{r-i}$ possibilities for picking $m-i$ women. Hence $\binom{n}{i}\binom{m}{r-i}$ gives us all possibilities for picking i men and $r-i$ women. Summing over all possible i gives the desired formula.

(8) In a hand of bridge find the probability that you have 5 spades, and your partner has the remaining 8 spades.

Solution: There are $\binom{52}{13.13,13,13}$ total number of bridge hands. I have $\binom{13}{5}$ choices to pick 5 spades. Since the partner gets the remaining spades, the 39 non spade cards get divided as follows, 8 for me, 5 for partner, 13 and 13 for the opposition. Hence there are $\binom{13}{5}\binom{39}{8,5,13,13}$ many possibilities. Divide by the total number of possible bridge hands to get the probability:

$$\frac{\binom{13}{5}\binom{39}{8,5,13,13}}{\binom{52}{13.13,13,13}}$$

(9) Compute the probability that a bridge hand is void in at least one suit.

Hint: Use Proposition 4.4.

Solution: Discussed at length in class.

(10) Prove Bonferroni's inequality:

$$P(AB) \geq P(A) + P(B) - 1.$$

Solution: $1 \geq P(A \cup B) = P(A) + P(B) - P(AB)$. Hence $P(AB) \geq P(A) + P(B) - 1$.