A BONUS TOPIC, MATRICES WITH NO LOGARITHM

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Problem 10 shows that any matrix which has a (real) logarithm has positive determinant, so we can deduce that $\left[\, -1 \,\right]$ or $\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ don't have logarithms. The slick proof also shows that a matrix which has a logarithm has a square root, so we can find other matrices without logarithms by finding matrices with out square roots. In particular, we claim that $\left[\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right]$ does not have a square root.

Proof: Suppose

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Looking at the upper right, $1 = ab + bd = b(a+d)$ so $a + d \neq 0$. Looking at the lower left, $0 = ca + dc = c(a+d)$ so $c = 0$. Then the upper left shows $a^2 = -1$, a contradiction. $\square$

Note that this shows that the set of matrices with logarithms is not a group, since

$$\exp \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix} \exp \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$