The Lie Bracket

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Problem 27

Let $X \subset \mathbb{R}^n$ be a manifold and $x \in X$. Suppose that $\gamma : (-\delta, \delta) \to X$ is a smooth path with $\gamma(0) = x, \gamma'(0) = 0$, and $\gamma''(0) = \vec{v}$. Show that $\vec{v} \in T_x X$. Check that this does not hold without the hypothesis $\gamma'(0) = 0$.

Proof [1]

Consider the function $\gamma_1 : (-\delta^2, \delta^2) \to X$ defined by

$$\gamma_1(t) = \gamma(\sqrt{t}) \text{ if } t \geq 0$$

$$\gamma_1(t) = \gamma_1(-t) \text{ if } t < 0$$

We have $\gamma_1(0) = \gamma(0) = 0$, and $\gamma_1'(0) = \lim\limits_{t \to 0} \frac{\gamma_1(t) - \gamma_1(0)}{t - 0}$

If the limit does exist, then we have $\lim\limits_{t \to 0} \frac{\gamma_1(t) - \gamma_1(0)}{t - 0} = \lim\limits_{t \to 0^+} \frac{\gamma_1(t) - \gamma_1(0)}{t - 0} = \lim\limits_{t \to 0^+} \frac{\gamma(\sqrt{t})}{t} = \lim\limits_{t \to 0^+} \frac{\gamma'(\sqrt{t})}{2\sqrt{t}} = \lim\limits_{s \to 0^+} \frac{\gamma'(s)}{2s} = \frac{\gamma''(0)}{2} = \frac{\vec{v}}{2}$. So we have $\frac{\vec{v}}{2} \in T_x X$. But $T_x X$ is a vector space, so we also have $\vec{v} \in T_x X$.

Remarks

The argument above crucially relies on the assumption that $\lim\limits_{t \to 0} \frac{\gamma_1(t) - \gamma_1(0)}{t - 0}$ exists. But the limit does not exist: $\lim\limits_{t \to 0^-} \frac{\gamma_1(t) - \gamma_1(0)}{t - 0} = \lim\limits_{t \to 0^-} \frac{\gamma(\sqrt{-t})}{t} = -\lim\limits_{t \to 0^-} \frac{\gamma(\sqrt{-t})}{-t} = -\lim\limits_{s \to 0^+} \frac{\gamma(\sqrt{s})}{s} = -\frac{\vec{v}}{2}$.

Another proof [2]

Suppose X is a d-fold, then by definition there is an open set $U \subset \mathbb{R}^n$ with$x \in U$, an open set

[1] Presented by Shiliang Gao in lecture
[2] Due to Nelson Zhang

$P \subset \mathbb{R}^d$, and a bijection $f : P \to X \cap U$, such that f is a smooth immersion, and f^{-1} is continuous.

Since γ is smooth, U is an open set around x, $\gamma(0) = x$, we can shrink δ, so that $\gamma(t) \in X \cap U$ for all $t \in (-\delta, \delta)$.

Consider the function $g : (-\delta, \delta) \to P$, defined by $g(t) = f^{-1}(\gamma(t))$. We have $\gamma(t) = f(g(t))$.

Then $0 = \gamma'(0) = Df_{g(0)}(g'(0))$. Let $p = f^{-1}(x)$. Then $g(0) = p$.

Since f is a immersion, Df_p is injective, $Df_p(g'(0)) = 0$, we must have $g'(0) = 0$

Then $\vec{v} = \gamma''(0) = (D(Df_{g(0)}))_{g'(0)}(g'(0)) + Df_{g(0)}(g''(0)) = 0 + Df_p(g''(0)) \in Df_p(\mathbb{R}^d) = T_xX$

Problem 28

Let $G \subset GL_n(\mathbb{R})$ be a Lie group, meaning G is simultaneously a subgroup of $GL_n(\mathbb{R})$ and a manifold in $\mathbb{R}^{n^2}$. Let $X, Y \in \mathfrak{g}$ and define $[X, Y] = XY - YX$. Show that $[X, Y] \in \mathfrak{g}$. Hint: Consider $e^{tX}e^{tY}e^{-tX}e^{-tY}$.

Proof [3]

Since $\mathfrak{g}$ is a vector space, closed under scalar multiplication, $tX, tY \in \mathfrak{g}$ for all $t \in \mathbb{R}$. Recall from Problem 24 that $\exp(\mathfrak{g}) \subseteq G$. So $\exp(tX) \in G$.

Since G is a group, note that $e^{tX}, e^{tY} \in G$. Similarly, $e^{tX}e^{tY}e^{-tX}e^{-tY} \in G$.

$$(\sum_a \frac{t^a X^a}{a!})(\sum_b \frac{t^b Y^b}{b!})(\sum_c \frac{(-1)^c t X^c}{c!})(\sum_d \frac{(-1)^d t X^d}{d!}) =$$

$$(\sum_{k=1}^{\infty} t^k \sum_{l=0}^{k} \frac{X^l Y^{k-1}}{(k-1)!(l!)})(\sum_{m=0}^{\infty} (-1)^m t^m)(\sum_{n=0}^{m} \frac{X^n Y^{m-n}}{(m-n)!(n!)}) =$$

$$\sum_{k=0}^{\infty} t^k \sum_{t=0}^{k} (-1)^{k-l} (\sum_{m=0}^{l} \frac{X^m Y^{l-m}}{(l-m)!(m!)})(\sum_{m=0}^{k-l} \frac{X^m Y^{k-l-m}}{(k-l-m)!(m!)})$$

By Problem 27, we have that the second derivative of the above expression should be in the tangent space, in this case, $\mathfrak{g}$.

$$D(\sum_{k=0}^{\infty} kt^{k-1} \sum_{t=0}^{k} (-1)^{k-l} (\sum_{m=0}^{l} \frac{X^m Y^{l-m}}{(l-m)!(m!)})(\sum_{m=0}^{k-l} \frac{X^m Y^{k-l-m}}{(k-l-m)!(m!)})) =$$

[3] Presented by Charlie Devlin in lecture

$$(\sum_{k=0}^{\infty}(k)(k-1)t^{k-2}\sum_{t=0}^{k}(-1)^{k-l}(\sum_{m=0}^{l}\frac{X^mY^{l-m}}{(l-m)!(m!)})(\sum_{m=0}^{k-l}\frac{X^mY^{k-l-m}}{(k-l-m)!(m!)})) =$$

$$2(XY-YX)+(t(terms))+(t^2(terms))...$$

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At t=0, this expression simplifies to $2(XY-YX) \in \mathfrak{g}$, which is closed under scalar multiplication. Therefore $(XY-YX) \in \mathfrak{g}$.