Math 395 IBL

Jaeyoon Kim, Raymond Luo, and Jason Liu

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Let $G \subset GL_n\mathbb{R} \subset Mat_{n\times n}\mathbb{R} \cong \mathbb{R}^{n^2}$ be a closed subgroup and a sub-manifold.
Remark G is a lie subgroup of $GL_n\mathbb{R}$ (not yet defined)
We define $\mathfrak{g} = T_{Id}G$
One example we already had was $T_{Id}SO(n) = \mathfrak{SO}(n) = \{X : X + X^T = 0\}$

Problem 23. *Suppose that $\gamma(t)$ is a smooth curve in GL_n with $\gamma(0) = Id$ and $\gamma'(0) = X$. Show that $\lim_{n\to\infty} \gamma(1/n)^n = exp(X)$.*

Proof. From previous IBL, we have shown that the inverse function theorem implies that we can restrict exp as a map from U to V where both U and V are open, $0 \in U$ and $Id \in V$ s.t. we have the inverse function denoted log. Additionally, since exp is smooth, by shrinking U and V sufficiently small, we can make log also smooth.

Since γ is a smooth function, for sufficiently small $t \in \mathbb{R}$, $\gamma(t) \in V$ Define $\delta(t) = log(\gamma(t))$. Since log is the inverse, we have $\gamma(t) = exp(\delta(t))$.

Since both γ and log are smooth, their composition δ is also smooth. Note that $\delta(0) = log(\gamma(0)) = log(Id) = 0$

Remember that $(Dexp)_0 = Id$ thus $(Dlog)_{Id} = (Dexp)_0^{-1} = Id$

$$\delta'(0) = (Dlog)_{\gamma(0)}\gamma'(0) = (Dlog)_{Id}X = Id\ X = X$$

$$\lim_{n\to\infty} \gamma(1/n)^n = \lim_{n\to\infty} [exp(\delta(1/n)]^n$$

Since $\delta(1/n)$ commutes with itself, $exp(\delta(1/n)^n = exp(n\delta(1/n))$

And by ~~L'Hopitol's Rule~~: definition of a derivative, $\lim_{t\to 0} \frac{\delta(t)}{t} = \delta'(0)$ Thus

$$\lim_{n\to\infty} \gamma(1/n)^n = \lim_{n\to\infty} exp(n\delta(1/n)) = exp(\lim_{t\to 0} \frac{\delta(t)}{t}) = exp(\delta'(0)) = exp(X)$$

□

Problem 24. *Show that $exp(\mathfrak{g}) \subset G$*

Proof. Fix $g \in \mathfrak{g}$. By definition of a tangent space, we can find $\gamma : (-\delta, \delta) \to G$ s.t. $\gamma(0) = Id$ and $\gamma'(0) = g$.

Since G is a group, it is closed under multiplication. Thus $\forall t \in (-\delta, \delta)$ and $\forall n \in \mathbb{N}$, $\gamma(t)^n \in G$. From problem 23, we know that $\lim_{n\to\infty} \gamma(1/n)^n = exp(g)$. Well since $\gamma(1/n)^n$ is a converge sequence living in G, a closed set, we know that its limit $exp(g)$ is also in G. □