A group that is Lie

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Problem 16: Show that there is an open set U containing Id_n such that $U \cap O(n)$ is a manifold.

Proof. As we showed in (13), there exist neighborhoods U containing 0 and V containing Id_n such that exp is a bijection from U to V, with smooth inverse log. In (15), we shrink U to U' and V to V', with $0 \in U'$ and $\text{Id}_n \in V'$, such that $V' \cap O(n) \subset \exp(\mathfrak{so}(n))$. Moreover, exp is bijective from U' to V' and $\exp(\mathfrak{so}(n)) \subset O(n)$.

We claim that $V' \cap O(n)$ is a manifold. To show this, set $P = \log(V' \cap O(n))$ and $\exp : P \to \mathbb{R}^{n^2}$. Notice that $P = \log(V' \cap O(n)) = U' \cap \mathfrak{so}(n)$, by previous remarks. Since $U' \cap \mathfrak{so}(n)$ is open in the subspace toplogy, P is open. Additionally, exp is a homeomorphic C^1 immersion which sends P to $V' \cap O(n)$, as follows:

1. Homeomorphic: As defined above, $\exp : P \to \mathbb{R}^{n^2}$ is continuous and bijective with continuous inverse; hence it is homeomorphic.
2. C^1: Recall that exp is smooth.
3. Immersion: It suffices to check that the derivative of exp is injective at 0. Well, for matrix Y,

$$(D\exp)_0(Y) = \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \frac{0^j \cdot Y \cdot 0^{n-j-1}}{n!} = Y,$$

   which is clearly injective.

It follows that $f(P) = V' \cap O(n)$; thus we conclude that $V' \cap O(n)$ is indeed a manifold. $\square$

Problem 17: Compute $T_{\text{Id}_n} O(n)$.

Proof. First we check that $O(n)$ is a manifold. See (18). Denote the set of $n \times n$ matricies by M_n. If $O(n)$ is a manifold, condition (2) guarantees that there exists C^1 submersion $g : M_n \to \mathbb{R}^{n^2-d}$ such that some stuff holds. Now, by class (10/9/17), we have the following result:

Theorem 1. *For manifold X and corresponding submersion g,*

$$T_z X = \ker((Dg)_z)$$

holds for all $z \in X$.

Applying the theorem gives that

$$\begin{aligned} T_{\mathrm{Id}_n} O(n) &= \ker((\mathrm{D}g)_{\mathrm{Id}_n} \\ &= \{M \in M_n \mid ((\mathrm{D}g)_{\mathrm{Id}_n} M = 0\} \\ &= \{M \in M_n \mid M + M^t = 0\} \\ &= \mathfrak{so}(n). \end{aligned}$$

□

Problem 18: Show that $O(n)$ is a manifold.

We present two proofs, one that slightly generalizes (16) by shifting the set centered at the identity and another that uses condition (2) in a clever way.

Proof. The idea behind (16) is that we can go from $U' \cap \mathfrak{so}(n)$ into $V' \cap O(n)$ through the exp map. However, $\mathrm{Id}_n \in V'$, and we want to be able to move this anywhere we want in $O(n)$. Fix $g \in O(n)$.

Consider the map $\varphi : O(n) \to O(n)$ which sends matrix M to gM. Then the composition map $\varphi \circ \exp : U' \cap \mathfrak{so}(n) \to O(n)$ sends matrix J to $g \exp(J)$. One can check that this is indeed a homeomorphic C^1 immersion, because the set of orthogonal matricies is a group. Proceeding in a similar fashion to (16), we get that $O(n)$ is a manifold. □

Proof. We use condition (2). We claim that the function $g : M_n \to \mathbb{R}^{n^2-d}$ which sends matrix M to MM^t is a C^1 submersion. This function is C^1 because matrix multiplication and transposition is smooth. This is a submersion because for $X, Y \in M_n$,

$$(Dg)_X(Y) = YX^t + XY^t \implies (Dg)_{\mathrm{Id}_n}(Y) = Y + Y^t,$$

and so

$$(Dg)_{\mathrm{Id}_n}\Big(\frac{Y - Y^t}{2}\Big) = Y.$$

Finally, observe that for matrix $M \in O(n)$,

$$g^{-1}(g(M)) = g^{-1}(MM^t) = g^{-1}(\mathrm{Id}_n) = O(n),$$

by properties of the orthogonal group. □

Corollary 1. *The image of g is symmetric, so we can restrict the target space to only count the diagonal entries and above. Hence the dimension of $O(n)$ is $\frac{n(n-1)}{2}$.*

Corollary 2. *$O(n)$ is a Lie group.*