MATH 395 - The Matrix Exponential: Examples

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Problem 1. Compute $\exp\begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix}$ for real numbers s and t.

Proof.

$$\exp\begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix}^n = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} s^n & 0 \\ 0 & t^n \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{s^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{t^n}{n!} \end{bmatrix} = \begin{bmatrix} e^s & 0 \\ 0 & e^t \end{bmatrix}$$

□

Problem 2. Compute $\exp\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}$.

Proof. Since $\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for $k \geq 2$,

$$\exp\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}^0 + \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}$$

□

Problem 3. Let $X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Let s and t be real numbers. Compute $e^{sX}, e^{tY}, e^{sX}e^{tY}$ and e^{sX+tY}.

Proof.

$$e^{sX} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{tY} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{sX}e^{tY} = \begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & s & st \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{tY}e^{sX} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & s & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{sX+tY} = \exp \begin{bmatrix} 0 & s & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & s & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & st \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & s & \frac{st}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

□

Problem 4. Show that

$$\exp \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for any real number θ.

Proof.

$$\exp \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{bmatrix} + \frac{1}{3!} \begin{bmatrix} 0 & \theta^3 \\ -\theta^3 & 0 \end{bmatrix} + \frac{1}{4!} \begin{bmatrix} \theta^4 & 0 \\ 0 & \theta^4 \end{bmatrix} + \frac{1}{5!} \begin{bmatrix} 0 & -\theta^5 \\ \theta^5 & 0 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \dots & -(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \dots) \\ \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \dots & 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \dots \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

□

Problem 5. Show that $\exp(gXg^{-1}) = g\ \exp(X)g^{-1}$.

Proof. We first claim that if A_n and B are $m \times m$ matrices, and $\lim_{n\to\infty} A_n$ exists, then

$$\lim_{n\to\infty} (BA_n) = B \lim_{n\to\infty} A_n$$

Well, let

$$\lim_{n\to\infty} A_n = A = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ a_{21} & & \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mm} \end{bmatrix}$$

Then, $\lim_{n\to\infty}(A_n)_{ij} = a_{ij}$.

Since $(BA_n)_{ij} = \sum_{k=1}^{m} B_{ik}(A_n)_{kj}$,

$$\lim_{n\to\infty}(BA_n)_{ij} = \lim_{n\to\infty}\left(\sum_{k=1}^{m} B_{ik}(A_n)_{kj}\right) = \sum_{k=1}^{m} B_{ik}\left(\lim_{n\to\infty}(A_n)_{kj}\right) = \sum_{k=1}^{m} B_{ik}A_{kj} = (BA)_{ij}$$

Thus, $\lim_{n\to\infty}(BA_n) = B\lim_{n\to\infty} A_n$.

Now back to the main problem. Note that $(gXg^{-1})^n = gX^ng^{-1}$. So from the claim,

$$\exp(gXg^{-1}) = \sum_{n=0}^{\infty} \frac{gX^ng^{-1}}{n!} = g\left(\sum_{n=0}^{\infty} \frac{X^n}{n!}\right)g^{-1}$$

since $\sum_{n=0}^{\infty} \frac{X^n}{n!}$ exists.

Therefore, $\exp(gXg^{-1}) = g\ \exp(X)g^{-1}$. □

Problem 6. If A, B are $n \times n$ matrices and $AB = BA$, show that $e^Ae^B = e^Be^A = e^{A+B}$.

Proof. First, we prove that $e^Ae^B = e^Be^A$.

$$LHS = \left(\sum_{i=0}^{\infty} \frac{A^i}{i!}\right)\left(\sum_{j=0}^{\infty} \frac{B^j}{j!}\right) = \sum_{i,j\geq 0} \frac{A^iB^j}{i!j!}$$

Since A, B commute,

$$LHS = \sum_{i,j\geq 0} \frac{B^jA^i}{j!i!} = \left(\sum_{j=0}^{\infty} \frac{B^j}{j!}\right)\left(\sum_{i=0}^{\infty} \frac{A^i}{i!}\right) = e^Be^A = RHS$$

Next, we prove that $e^Ae^B = e^{A+B}$.

$$RHS = \sum_{n=0}^{\infty} \frac{\sum_{i=0}^{n}\binom{n}{i}A^iB^{n-i}}{n!} = \sum_{n=0}^{\infty}\sum_{i=0}^{n} \frac{A^iB^{n-i}}{(n-i)!i!} = \sum_{i,j\geq 0} \frac{A^iB^j}{i!j!} = LHS$$

□

Problem 7. Show that $e^Xe^{-X} = Id \implies e^X$ is invertible.

Proof. From Problem 6, we have:

$$e^Xe^{-X} = e^{X+(-X)} = e^0 = Id$$

Thus, any matrix that is an exponential is invertible. □