2. Let \( g(x, y) = \frac{x^3 y}{x^2 + y^2} \) for \((x, y) \neq (0, 0)\) and \( g(0, 0) = 0 \).

   (a) Show that, for any \( a \) and \( b \in \mathbb{R} \), the function \( g(at, bt) \) is differentiable, and \( \frac{dg(at, bt)}{dt} \bigg|_{t=0} = 0 \).
   (b) Show that \( g(x, y) \) is not differentiable.

3. Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a linear map with matrix \( A \). Recall that the **rank** of \( f \) is defined to be the dimension of the subspace \( f(\mathbb{R}^n) \) of \( \mathbb{R}^m \). The goal of this problem is to prove that \( \text{rank}(f) \geq k \) if and only if there is a \( k \times k \) submatrix of \( A \) with nonzero determinant.

   We introduce some helpful notations: For \( I \) any subset of \( \{1, 2, \ldots, m\} \) and \( J \) any subset of \( \{1, 2, \ldots, n\} \), let \( A_{IJ} \) denote the submatrix \( (A_{ij})_{(i,j)\in I \times J} \). We’ll write \( f_{IJ} \) for the linear map \( \mathbb{R}^{|I|} \to \mathbb{R}^{|J|} \) corresponding to \( A_{IJ} \).

   (a) Let \( |I| = |J| = k \). Show that \( \text{rank}(f_{IJ}) = k \) if and only if \( \text{det}(A_{IJ}) \neq 0 \).
   (b) Show that, for any \( I \) and \( J \), we have \( \text{rank}(f) \geq \text{rank}(f_{IJ}) \).
   (c) Let \( r = \text{rank}(f) \). Show that there is \( J \subset \{1, 2, \ldots, n\} \) with \( |J| = r \) and \( \text{rank}(f_{\{1,2,\ldots,m\},J}) = r \).
   (d) Let \( r = \text{rank}(f) \). Show that there is \( I \subset \{1, 2, \ldots, m\} \) and \( J \subset \{1, 2, \ldots, n\} \), both of size \( r \), so that \( \text{rank}(f_{IJ}) = r \).
   (e) Show that \( \text{rank}(f) \geq k \) if and only if there is a \( k \times k \) submatrix of \( A \) with nonzero determinant.

4. Let \( A \) be a linear map from \( \mathbb{R}^m \to \mathbb{R}^n \).

   (a) Suppose that \( m \leq n \) and \( \text{rank}(A) = m \). Show that we can find a linear map \( B : \mathbb{R}^{n-m} \to \mathbb{R}^n \) such that the map \( A \oplus B : \mathbb{R}^m \oplus \mathbb{R}^{n-m} \to \mathbb{R}^n \) is invertible. (Here \( (A \oplus B)(\vec{v} + \vec{w}) = A(\vec{v}) + B(\vec{w}) \) for \( \vec{v} \in \mathbb{R}^m \) and \( \vec{w} \in \mathbb{R}^{n-m} \).

   (b) Suppose that \( m \geq n \) and \( \text{rank}(A) = n \). Show that we can find a linear map \( B : \mathbb{R}^m \to \mathbb{R}^{m-n} \) such that the map \( A \oplus B : \mathbb{R}^m \to \mathbb{R}^{m-n} \oplus \mathbb{R}^n \) is invertible. (Here \( (A \oplus B)(\vec{v}) = A(\vec{v}) + B(\vec{\nu}) \). Note that \( A(\vec{v}) \in \mathbb{R}^m \) and \( B(\vec{\nu}) \in \mathbb{R}^{m-n} \).)

5. Let \( V \) and \( W \) be finite dimensional real vector spaces, let \( A \) be an open subset of \( V \) and let \( f : A \to W \) be a continuous function. The function \( f \) is called **proper** if, for any \( R > 0 \), the set \( \{ x \in A : |f(x)| \leq R \} \) is compact. (Recall that a subset of \( V \) is compact if it is closed in \( V \) and bounded.)

   (a) Show that the image of a proper map is closed.
   (b) Let \( g = g_n z^n + g_{n-1} z^{n-1} + \cdots + g_0 \) be a polynomial with complex coefficients and \( g_n \) nonzero. Let \( f(x, y) = (\text{Re}(g(x + iy)), \text{Im}(g(x + iy))) \). Show that \( f \) is proper.
   (c) The map \( f \) is called **open** if, for any open \( U \subset A \), the image \( f(U) \) is open. Show that, if \( f \) is proper and open, then \( f(A) = W \).

   If you suspect this is heading for a proof of the fundamental theorem of algebra, you are right. More next time.

   **Turn over for one more great question!**
6. This question proves that the matrix exponential is differentiable, with

$$(D \exp)_X(Y) = \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \frac{X^m Y X^{n-m-1}}{n!}.$$ 

I review definitions and results which I expect we will see in class on Friday the 15th: For a real matrix $X$, we define $|X| = \sqrt{\sum_{i,j} X_{ij}^2}$. I expect we will show, and you may use, that $|XY| \leq |X| \cdot |Y|$. I expect we will show, and you may use, that $|X + Y| \leq |X| + |Y|$. 

(a) Show that there exists a constant $C$ such that $|\exp(Y) - \text{Id} - Y| \leq C|Y|^2$ for all $Y$ with $|Y| < 1$.

(b) Deduce that $(D \exp)_0(Y) = Y$.

(c) Let $X$ be a $k \times k$ nonzero real matrix with $|X| = R > 0$ and let $Y$ be a $k \times k$ real matrix with $|Y| = r < R$. Show that

$$|(X + Y)^n - X^n - \sum_{m=0}^{n-1} X^m Y X^{n-m-1}| \leq 2^n R^{n-2} r^2.$$ 

(d) Fix $0 < r < R$. Show that there is a constant $C$ (dependent on $r$ and $R$) such that, if $|X| \in (r, R)$ and $|Y| < r$, we have

$$|\exp(X + Y) - \exp(X) - \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \frac{X^m Y X^{n-m-1}}{n!}| \leq C|Y|^2.$$ 

(e) Deduce that

$$(D \exp)_X(Y) = \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \frac{X^m Y X^{n-m-1}}{n!}.$$