Problem Set 2 – due September 29

See the course website for policy on collaboration.

1. Let $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 2y^3 + 3z^4 = 6, \ x^2 + y^2 + z^2 = 3\}$. 
Show that there is an interval $(1 - a, 1 + a)$ around 1 and smooth functions $y$ and $z : (1 - a, 1 + a) \to \mathbb{R}$ so that $y(1) = z(1) = 1$ and $(x, y(x), z(x))$ is in $C$ for all $x \in (1 - a, 1 + a)$.

2. Let $f(x, y) = y^3 - x^5 - x^3$. Is there a differentiable function $g : \mathbb{R} \to \mathbb{R}$ so that $g(0) = 0$ and $f(x, g(x)) = 0$?

3. Suppose that we have a differentiable function $(x, y) \mapsto (\alpha(x, y), \beta(x, y))$ from an open set in $\mathbb{R}^2$ to $\mathbb{R}^2$ so that 

$$e^x \cos \alpha(x, y) + e^y \cos \beta(x, y) = 1 \quad e^x \sin \alpha(x, y) + e^y \sin \beta(x, y) = 0.
$$

(a) Compute the matrix of partials $\left( \frac{\partial \alpha}{\partial x} \frac{\partial \alpha}{\partial y}, \frac{\partial \beta}{\partial x} \frac{\partial \beta}{\partial y} \right)$.

(b) Show that, if $e^{x_0} + e^{y_0} > 1$, $e^{x_0} + 1 > e^{y_0}$ and $e^{y_0} + 1 > e^{x_0}$, then there is an open set $U$ containing $(x_0, y_0)$ for which such functions $\alpha$ and $\beta$ exist.

4. Let $A \subset \mathbb{R}^n$ be an open set and $f : A \to \mathbb{R}$ a smooth function. Let $a \in A$.

(a) Suppose that $f(a) = \min_{x \in A} f(x)$. Show that $(Df)_a = 0$.

Let $x_1, \ldots, x_n$ be the coordinates on $\mathbb{R}^n$ and set $f_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_{ij}}(a)$.

(b) Suppose again that $f(a) = \min_{x \in A} f(x)$. Show that, for all $(v_1, \ldots, v_n) \in \mathbb{R}^n$ we have 

$$\sum_{i,j} f_{ij} v_i v_j \geq 0.$$

(c) Suppose that $(Df)_a = 0$ and, for all $(v_1, \ldots, v_n) \in \mathbb{R}^n \setminus \{(0, 0, \ldots, 0)\}$ we have 

$$\sum_{i,j} f_{ij} v_i v_j > 0.$$ 

Show that there is an open set $A' \ni a$ such that $f(a) = \min_{x \in A'} f(x)$.

5. This question is about considering polynomials in $\mathbb{C}[z]$ as differentiable maps $\mathbb{R}^2 \to \mathbb{R}^2$. For a function $g : \mathbb{C} \to \mathbb{C}$, we will write $\hat{g}$ for the corresponding function $\hat{g}(x, y) = (\text{Re} g(z + iy), \text{Im} g(z + iy))$, mapping $\mathbb{R}^2$ to $\mathbb{R}^2$. For $\alpha \in \mathbb{C}$, write $\mu_\alpha$ for the linear map $z \mapsto \alpha z$ from $\mathbb{C} \to \mathbb{C}$. So $\mu_\alpha$ is a linear map $\mathbb{R}^2 \to \mathbb{R}^2$.

(a) Show that $\mu_\alpha$ is invertible for $\alpha \neq 0$.

(b) For $g_n, g_{n-1}, \ldots, g_0 \in \mathbb{C}$, let $g(z) = \sum_{j=0}^{n} g_j z^j$ and define $g'(z) = \sum_{j=0}^{n} j g_j z^{j-1}$ (the formal derivative of $g$). Show that $(D\hat{g})(x, y) = \hat{\mu}_{g'(x+iy)}$.

(c) Let $U$ be open in $\mathbb{C}$ and let $g(z)$ be a nonzero polynomial in $\mathbb{C}[z]$. Suppose that $g'(z)$ is nonzero on $U$. Show that $g(U)$ is open in $\mathbb{C}$.

We will now work to prove that $g(U)$ is open without the hypothesis that $g'(z) \neq 0$.

(d) Suppose that $g(0) = 0$ but $g$ is not identically 0. Let $k$ be the minimal index for which $g_k \neq 0$. Show that there is an open set $V \ni 0$ on which we can write $g(z) = r(z)^k$ with $r : V \to \mathbb{C}$ smooth and $(D\hat{r})_0$ invertible.

(e) In the above notation, show that there is an $\epsilon > 0$ such that $g(V)$ contains the disc of radius $\epsilon$ around 0.

(f) Let $g$ be any nonzero polynomial in $\mathbb{C}[z]$. Show that, if $U$ is open, then $g(U)$ is open.
6. This question addresses the following question: Let $X$ and $Y$ be topological spaces and let $f : X \times Y \to \mathbb{R}$ be a continuous function. Set $g(x) = \inf_{y \in Y} f(x, y)$. Is $g$ continuous?

(a) Define $f(x, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $f(x, y) = \exp(-x^2 y^2)$. Define $g(x) = \inf_{y \in Y} f(x, y)$. Compute $g$ and show that it is NOT continuous.

(b) Suppose that $Y$ is compact (if it would make you more comfortable, you may also assume $X$ and $Y$ are metric spaces). Show then that $g$ is continuous.

(c) In (a) there was not actually a point where the infimum $\inf_{y \in Y} f(x, y)$ was achieved (for $x \neq 0$). If we do not require that $Y$ is compact, but do require that, for all $x$ there is a $y_x$ such that $f(x, y_x) \leq f(x, y)$ for all $y \in Y$, will $g$ necessarily be continuous?