Continuity and differentiability of the exponential

Given an $n \times n$ real matrix A_{ij}, we define $|A_{ij}| = \sqrt{\sum_{i,j} A_{ij}^2}$.

Problem 8. Show that
$$|AB| \leq |A| \cdot |B|.$$
Hint: Remember the Cauchy-Schwartz inequality: $\left(\sum_{k=1}^n c_k d_k\right)^2 \leq \sum_{k=1}^n c_k^2 \sum_{\ell=1}^n d_\ell^2$.

We recall the Weierstrass M-test: If $f_n(X)$ is a sequence of continuous functions of a variable X (in $\mathbb{R}^k$, say) and M_n is a sequence of positive numbers such that $|f_n(X)| \leq M_n$ and $\sum_{n=0}^\infty M_n < \infty$, then $\sum_{n=0}^\infty f_n(X)$ is convergent and converges to a continuous function.

Problem 9. Let R be a positive real number and let $B(R) = \{X \in \mathrm{Mat}_{n\times n}(\mathbb{R}) : |X| \leq R\}$. Show that $\exp : B(R) \to \mathrm{GL}_n(\mathbb{R})$ is continuous. Deduce that exp is continuous.

Problem 10. Show that is X is an $n \times n$ real matrix then $\det \exp(X) > 0$.

We now consider differentiability of exp.

Problem 11. For X a $k \times k$ matrix, let $g(X) = X^n$. For any $k \times k$ matrix Y, show that $D(g)_X(Y) = \sum_{j=0}^{n-1} X^j Y X^{n-1-j}$.

On the next problem set, you'll show that it is legitimate to differentiate the sum defining exp term by term, giving:
$$(D\exp)_X(Y) = \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \frac{X^j Y X^{n-1-j}}{n!}.$$