The matrix exponential : Examples

Given an $n \times n$ matrix X, we define $e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$. We will also write $\exp(X)$.

Problem 1. Compute $\exp\left[\begin{smallmatrix} s & 0 \\ 0 & t \end{smallmatrix}\right]$ for real numbers s and t.

Problem 2. Compute $\exp\left[\begin{smallmatrix} 0 & u \\ 0 & 0 \end{smallmatrix}\right]$.

Problem 3. Let $X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Let s and t be real numbers. Compute e^{sX}, e^{tY}, $e^{sX}e^{tY}$, $e^{tY}e^{sX}$ and e^{sX+tY}.

Problem 4. Show that

$$exp\left[\begin{smallmatrix} 0 & -\theta \\ \theta & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$$

for any real number θ.

Problem 5. Show that $\exp(gXg^{-1}) = g\exp(X)g^{-1}$.

Problem 6. (added in class) Show that, if A and B commute,

$$\exp(A)\exp(B) = \exp(B)\exp(A) = \exp(A+B).$$

Problem 7. (added in class) Show that $\exp(X)\exp(-X) = \mathrm{Id}$. Deduce that $\det\exp(X)$ is always $\neq 0$.