Problem Set 2 – due Tuesday January 27

See the course website for policy on collaboration.

**Problem 1** Consider the vector space $\mathbb{R}^3$ with its standard inner product $\cdot$. Let $\text{Vol}$ be the element $e_1 \wedge e_2 \wedge e_3$ in $\bigwedge^3 \mathbb{R}^3$, where $(e_1, e_2, e_3)$ is the standard basis of $\mathbb{R}^3$. For $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$, show that there is a unique element $c(\vec{u}, \vec{v})$ of $\mathbb{R}^3$ such that, for all $\vec{w} \in \mathbb{R}^3$, we have

$$\vec{u} \wedge \vec{v} \wedge \vec{w} = (\vec{u} \cdot c(\vec{u}, \vec{v})) \text{Vol}.$$

What is the standard name for the function $c(\vec{u}, \vec{v})$?

**Problem 2** Let $A$ be an $m \times n$ matrix, which we also consider as a linear map $\mathbb{R}^n \to \mathbb{R}^m$. Let $k$ be a positive integer. Recall that the rank of $A$ is defined to be the dimension of the image of $A$. In this problem, we return to a problem from the zeroeth problem set of Fall term, where we showed that $\text{rank}(A) \geq k$ if and only if $A$ has a $k \times k$ submatrix with nonzero determinant.

(a) Show that $A$ has a $k \times k$ matrix with nonzero determinant if and only if $\bigwedge^k A$ is nonzero.

(b) Suppose that $\text{rank}(A) < k$. Show that $\bigwedge^k A = 0$. (Hint: The nice solution uses functoriality.)

(c) Suppose that $\text{rank}(A) \geq k$. Show that $\bigwedge^k A \neq 0$. (Hint: Take $k$ elements of $\mathbb{R}^n$ that have linear independent images under $A$, and build an element of $\bigwedge^k \mathbb{R}^n$ with nonzero image.)

**Problem 3.** (a) Let $B$ be a nilpotent $n \times n$ matrix with entries in a field $k$, meaning that $B^N = 0$ for some positive integer $N$. Show that $cId_n + B$ is invertible for any nonzero scalar $c$. (Hint: geometric series.)

(b) Let $r(x)$ be a polynomial with entries in $k$ and $r(0) = 0$. Show that $r(B)$ is nilpotent.

(c) Let $r(x)$ be a polynomial with entries in $k$ and $r(0) \neq 0$. Show that $r(B)$ is invertible.

**Problem 4** Recall from class that we defined $C(k, n)$ to be the set of tensors in $\bigwedge^k \mathbb{R}^n$ which are of the form $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ for some $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$; we write $e_1, e_2, \ldots, e_n$ for the standard basis of $\mathbb{R}^n$. We defined $OG(k, n)$ to be the intersection of $C(k, n)$ with the unit sphere $S^{(k)}(1)$ in $\mathbb{R}^{(k)}$. In this problem, we will check that $OG(k, n)$ is a submanifold of $\bigwedge^k \mathbb{R}^n$.

(a) Suppose that $\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} p_{i_1 i_2 \cdots i_k} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$ is a point in $C(k, n)$ with $p_{12 \cdots k} = 1$. Show that $\omega$ can be written in the form $v_1 \wedge \cdots \wedge v_k$ where $v_j$ is of the form $e_j + \sum_{r=1}^{n-k} A_{jr} e_{k+r}$ for some $k \times (n-k)$ matrix $A$.

(b) In the notation of part (a), show that the entries of $A$ are uniquely determined by $\omega$.

(c) Give an injective immersion $\tilde{\phi} : \mathbb{R}^{k(n-k)} \to \bigwedge^k \mathbb{R}^n$ whose image is $C(k, n) \cap \{p_{12\cdots k} = 1\}$.

(d) Give an injective immersion $\phi : \mathbb{R}^{k(n-k)} \to \bigwedge^k \mathbb{R}^n$ whose image is $OG(k, n) \cap \{p_{12\cdots k} > 0\}$. (Hint: Modify $\tilde{\phi}$.)

(e) Show that $OG(k, n)$ can be covered by open sets $U$ such that, for each $U$, the intersection $OG(k, n) \cap U$ is the image of an injective immersion $\mathbb{R}^{k(n-k)} \to OG(k, n) \cap U$. In other words, $OG(k, n)$ is a submanifold of $\bigwedge^k \mathbb{R}^n$, of dimension $\mathbb{R}^n$. 