Problem Set 3 – due Tuesday February 10

See the course website for policy on collaboration.

Problem 1 Let $V$ be a finite dimensional $\mathbb{R}$ vector space. Let $\omega$ be a function from $V \to \bigwedge^p V^*$. Let $x_1, x_2, \ldots, x_n$ be coordinates on $V$ and write

$$\omega_a = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} f_{i_1 \cdots i_p}(a) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}.$$ 

Show that the following are equivalent:

(i) $\omega$ is a smooth map.
(ii) For any vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $V$, the function $a \mapsto \omega_a(\vec{v}_1, \ldots, \vec{v}_p)$ is smooth.
(iii) The functions $f_{i_1 \cdots i_p}$ are smooth.

Problem 2 Let $V$ be a finite dimensional real vector space. Let $V^*$ be the dual space and let $(\ , \ )$ be the bilinear pairing between $V^*$ and $V$. Let $(\ , \ ) : V \otimes V \otimes \cdots \otimes V \times V^* \otimes V^* \otimes \cdots \otimes V^*$ be the pairing such that $(v_1 \otimes v_2 \otimes \cdots \otimes v_p, w_1 \otimes w_2 \otimes \cdots \otimes w_p) = \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle \cdots \langle v_p, w_p \rangle$ for any $v_1, \ldots, v_p \in V$ and $w_1, \ldots, w_p \in V^*$. You may assume this pairing exists.

(a) Show that $(\ , \ )$ is a perfect pairing. (In other words, there are bases for $V \otimes \cdots \otimes V$ and $V^* \otimes \cdots \otimes V^*$ for which the matrix of $(\ , \ )$ is the identity matrix.)
(b) Show that the restriction of $(\ , \ )$ to $\bigwedge^p V$ and $\bigwedge^p V^*$ is perfect.
We can therefore identify $(V \otimes \cdots \otimes V)^*$ and $V^* \otimes \cdots \otimes V^*$, and identify $\bigwedge^p (V^*)$ and $\bigwedge^p V^*$.

Problem 3 We repeat a problem from the midterm: Let $X(k, n)$ be the space of $n \times n$ symmetric matrices $B$ satisfying $B^2 = B$, $B^T = B$ and $\text{Tr}(B) = k$. In this problem, we will show that $X(k, n)$ is a submanifold of the $n \times n$ matrices.

Let $M$ be the set of $k \times n$ matrices of rank $k$. For $A \in M$, define $\phi(A) = A^T (AA^T)^{-1} A$. (From Problem Set 1, Problem 4(a), the matrix $AA^T$ is invertible; you may assume this.)

(a) Show that $\phi(A)$ lies in $X(k, n)$.
(b) Let $B \in X(k, n)$, let $V$ be the image of $B$ and let $A$ be a $k \times n$ matrix with row span $V$. Show that $B = \phi(A)$. (Hint: Check $B\vec{v} = \phi(A)\vec{v}$ if $\vec{v}$ is in $V$, and if $\vec{v}$ is in $V^\perp$.)

We now know that $\phi(M) = X(k, n)$.

For any $n \times n$ matrix $B$, write $B$ in block form as \[
\begin{pmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{pmatrix}
\] where $B_{11}$ is a $k \times k$ matrix and so forth. Let $U$ be the set of $n \times n$ matrices for which $\det B_{11} \neq 0$. Define a map $\psi$ from $U$ to $k \times (n - k)$ matrices by $\psi(B) = (B_{11})^{-1} B_{12}$.

(c) Show that $\psi(\phi((\text{Id}_k \ E))) = E$. Here $(\text{Id}_k \ E)$ is a $k \times n$ matrix, made up of a $k \times k$ block and a $k \times (n - k)$ block.

(d) Show that $E \mapsto \phi((\text{Id}_k \ E))$ is an injective immersion from $\mathbb{R}^{k(n-k)}$ to $\mathbb{R}^{n^2}$, with image $X(k, n) \cap U$.

(e) Show that we can cover $X(k, n)$ with open sets $U_i$ so that, for each $U_i$, there is an injective immersion $\phi_i : \mathbb{R}^{k(n-k)} \to U_i$ with $X(k, n) \cap U_i = \phi_i(\mathbb{R}^{k(n-k)})$.

In other words, $X(k, n)$ is a $k(n - k)$-dimensional submanifold of $\mathbb{R}^{n^2}$. 