Problem 1 This repeats an example done in class; please do work it out. Let \( \phi: (0, \infty) \times \mathbb{R} \to \mathbb{R}^2 \setminus \{(0,0)\} \) be the map \( \phi(r, \theta) = (r \cos \theta, r \sin \theta) \). Compute:

(a) \( \phi^*(dx) \).

(b) \( \phi^* \frac{x dy - y dx}{x^2 + y^2} \).

(c) \( \phi^*(dx \wedge dy) \).

Problem 2 Let \( V \) be a three dimensional real vector space equipped with a (symmetric, positive definite) inner product \( \cdot \).

(a) Let \( (e_1, e_2, e_3) \) and \( (f_1, f_2, f_3) \) be two orthonormal bases of \( V \). Show that \( e_1 \wedge e_2 \wedge e_3 = \pm f_1 \wedge f_2 \wedge f_3 \).

We fix once and for all an element \( \text{Vol} \) which is of the form \( e_1 \wedge e_2 \wedge e_3 \) from some orthonormal basis \( (e_1, e_2, e_3) \) of \( V \). Let \( x_1, x_2, x_3 \) be the basis of \( V^* \) dual to \( e_1, e_2, e_3 \), meaning that \( x_i(e_j) = 1 \) if \( i = j \) and \( 0 \) otherwise.

(b) Define a map \( a: V \to V^* \) by \( a(v)(w) = v \cdot w \). Show that \( a \) is invertible. Write \( a(e_i) \) in the basis \( x_i \). (The answer is very simple.)

(c) Let \( A \) be an open set in \( V \). Let \( f \) be a smooth real valued function on \( A \). Define \( \nabla f : A \to V \) to be the composition of \( df : A \to V^* \) and \( a^{-1} : V^* \to V \). If we write \( \nabla f = g_1 e_1 + g_2 e_2 + g_3 e_3 \), what are the \( g_i \) in terms of derivatives of \( f \)?

(d) Define a map \( h : \wedge^2 V \to V \) so that, for \( v \in V \) and \( \omega \in \wedge^2 V \), we have \( v \wedge \omega = (v \cdot h(\omega)) \text{Vol} \). Show that \( h \) is invertible, and compute \( h(e_1 \wedge e_2), h(e_1 \wedge e_3) \) and \( h(e_2 \wedge e_3) \) in the basis \( e_1, e_2, e_3 \).

(e) Let \( u \) be a smooth map \( A \to V \); we write \( u = u_1 e_1 + u_2 e_2 + u_3 e_3 \). Then \( a \circ u \) is a 1-form (sending \( A \to V^* \)) and \( d(a \circ u) \) is a 2-form (sending \( A \to \wedge^2 V^* \)). Then \( h \circ \wedge^2 a^{-1} \circ d(a \circ u) \) is a smooth map \( A \to V \). We define \( \nabla \times u \) to be \( h \circ \wedge^2 a^{-1} \circ d(a \circ u) \). Writing \( \nabla \times u = v_1 e_1 + v_2 e_2 + v_3 e_3 \), what are the \( v_i \) in terms of derivatives of the \( u_i \)?

Note: \( \nabla f \), \( \nabla \times u \) and \( \nabla \cdot v \) are called the gradient, curl and divergence of \( f, u \) and \( v \) respectively. Feel free to look up the standard formulas for these operators; they should match your results here.

Problem 3 In this problem, we will prove Poincare’s lemma: Closed forms on \( \mathbb{R}^n \) are also exact. This is the most direct proof I know, though not the most enlightening.

Define \( \Omega^p_m(\mathbb{R}^n) \) to be those smooth differential forms of the form

\[
\sum_{1 \leq i_1 < i_2 < \ldots < i_p \leq m} f_{i_1i_2\ldots i_p}(x_1, \ldots, x_n)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}.
\]

(The variable \( m \) appears in the bounds of the sum.) We will prove the following:

Claim: If \( \omega \in \Omega^p_m(\mathbb{R}^n) \) and \( d\omega = 0 \), then \( \omega \) is of the form \( d\eta \).

(a) Explain why the Claim is trivially true for \( m < p \). This is our base case.
(b) Let $\omega \in \Omega^p_m(\mathbb{R}^n)$ with $d\omega = 0$ and write $\omega$ in the form

$$
\sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq m-1} g_{i_1 i_2 \cdots i_p}(x_1, \ldots, x_n)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} \\
+ \sum_{1 \leq j_1 < j_2 < \cdots < j_{p-1} \leq m-1} h_{j_1 j_2 \cdots j_{p-1} m}(x_1, \ldots, x_n)dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_{p-1}} \wedge dx_m.
$$

Show that the functions $h_{j_1 j_2 \cdots j_{p-1} m}(x_1, \ldots, x_n)$ don't depend on $x_{m+1}, x_{m+2}, \ldots, x_n$.

(c) Define

$$
r_{j_1 \cdots j_{p-1}}(x_1, \ldots, x_n) = \int_{u=0}^{x_m} h_{j_1 j_2 \cdots j_{p-1} m}(x_1, \ldots, x_{m-1}, u, x_{m+1}, \ldots, x_n)du
$$

and

$$
\beta = (-1)^{p-1} \sum_{1 \leq j_1 < j_2 < \cdots < j_{p-1} \leq m-1} r_{j_1 \cdots j_{p-1}}(x_1, \ldots, x_n)dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_{p-1}}.
$$

Show that $\omega - d\beta \in \Omega^p_{m-1}(\mathbb{R}^n)$. (Make sure to explain why there are no $dx_a$ terms for $a > m$, as well as why there are no $dx_m$ terms.)

(d) Show that the Claim for $m - 1$ implies the Claim for $m$, and thus complete the proof.