Problem Set 5 – due Tuesday March 17

See the course website for policy on collaboration.

Problem 1 Remember that a topological space is a set $X$ equipped with a collection of subsets called \textbf{open subsets} such that:

1. $\emptyset$ and $X$ are open
2. If $U$ and $V$ are open, then $U \cap V$ is open
3. If $U_i$ is any collection of open subsets (for $i$ running through any index set $I$) then $\bigcup U_i$ is open.

Let $X$ be a topology and let $\sim$ be an equivalence relation on $X$. Let $X/\sim$ be the set of equivalence classes for $\sim$ on $X$, and let $\pi : X \to X/\sim$ be the map sending a point to its equivalence class. Define a topology on $Y$ two topological spaces $S$ embedded manifolds, you can think of $f$ by $S^\nu_p$ for every $p \in S$. Let $\rho$ be the number of points where this map is orientation reversing. In this problem, we will show that $n$ be the number of point $x$ you like. Let $\phi$ have $\delta$.

(a) Check that this is a topology.
(b) Let $\sim$ be the equivalence relation on $[0, 2\pi]$ where $0 \sim 2\pi$, every point is equivalent to itself, and there are no other relations. Let $S^1$ be $\{(x, y) : x^2 + y^2 = 1\}$. Define a map $f : [0, 2\pi] \sim \to S^1$ by $f(\theta) = (\cos \theta, \sin \theta)$. Show that $f$ and $f^{-1}$ are continuous. (Recall a map $g : Y \to Z$ between two topological spaces $Y$ and $Z$ is called continuous if, for any open $U \subset Y$, the preimage $g^{-1}(U)$ is open.)

Problem 2 Let $S^1$ denote the unit circle and let $T = S^1 \times S^1$. If you still are thinking in embedded manifolds, you can think of $S^1$ as embedded in $\mathbb{R}^2$ and $T$ as embedded in $\mathbb{R}^4$. Fix a point $p$ in $S^1$. Let $X$ and $Y$ be the curves $S^1 \times \{p\}$ and $\{p\} \times S^1$ in $Y$. We choose orientations $\nu_X$ and $\nu_Y$ of $X$ and $Y$, and take the orientation $\nu_X \wedge \nu_Y$ of $T$.

Let $\alpha$ and $\beta$ be \textbf{closed} 1-forms on $T$. In this problem, we will establish the identity

$$\int_T (\alpha \wedge \beta) = \int_X \alpha \int_Y \beta - \int_X \beta \int_Y \alpha.$$  

(a) Explain why $\int_X \alpha$ doesn’t depend on the choice of the point $p$.
(b) Let $Q = [0, 2\pi] \times [0, 2\pi]$ and let $\rho : Q \to T$ send $(\theta, \phi)$ to $(\cos \theta, \sin \theta) \times (\cos \phi, \sin \phi)$. We take $X = \rho ([0, 2\pi] \times \{0\})$ and $Y = \rho (\{0\} \times [0, 2\pi])$.
(c) Explain why $\rho^* \alpha$ and $\rho^* \beta$ are closed on $Q$. Explain by which previous result we know that $\rho^* \alpha$ and $\rho^* \beta$ are exact on $Q$.
(d) Let $\rho^* \alpha = df$ for a function $f$ on $Q$.
(e) Show that, for any $y \in [0, 2\pi]$, we have $f(2\pi, y) - f(0, y) = \int_X \alpha$ and, for any $x \in [0, 2\pi]$, we have $f(x, 2\pi) - f(x, 0) = \int_Y \alpha$.
(f) Show that $\rho^*(\alpha \wedge \beta) = d(f \cdot \rho^* \beta)$.
(g) Show that

$$\int_T (\alpha \wedge \beta) = \int_X \alpha \int_Y \beta - \int_X \beta \int_Y \alpha.$$  

Problem 3 Let $X$ be a compact oriented 2-fold. You can think of it embedded in some $\mathbb{R}^N$ if you like. Let $\phi : X \to \mathbb{R}^2$ be a smooth map. Let $q$ be a point in $\mathbb{R}^2$ such that $\phi^{-1}(q)$ is finite and, for every $p \in \phi^{-1}(q)$, the map $(D\phi)_p$ is invertible. We’ll write $\phi^{-1}(q) = \{p_1, p_2, \ldots, p_n\}$. Let $n_+$ be the number of point $p_i$ for which $(D\phi)_p$ is orientation preserving, and let $n_-$ be the number of points where this map is orientation reversing. In this problem, we will show that $n_+ = n_-$. We will choose coordinates $x$ and $y$ on $\mathbb{R}^2$ so that $q$ is at $(0, 0)$. We write $B(q, \delta)$ for the ball of radius $\delta$ around $q$.

(a) Show that, for $\delta$ sufficiently small, there are disjoint open sets $U_1, U_2, \ldots, U_n$ in $X$, with $p_i \in U_i$, for which $\phi$ is a diffeomorphism $U_i \to B(q, \delta)$. (Hint: Pass to a patch around each $p_i$, and recall the theorem we already have for this purpose.)
Let $h : [0, \infty) \to \mathbb{R}_{\geq 0}$ be a smooth function with $h(t) = 1$ for $t \in [0, 1/2]$ and $h(t) = 0$ for $t \geq 1$. Define $\omega$ to be the 2-form $h \left( \frac{x^2 + y^2}{\delta^2} \right) dx \wedge dy$ and set $A = \int_{\mathbb{R}^2} \omega$.

(b) Show that $\int_X \phi^* \omega = (n_+ - n_-)A$.

(c) Explain what previous result shows that $\omega$ is exact on $\mathbb{R}^2$. Show that $\phi^* \omega$ is exact on $X$.

(d) Show that $\int_X \phi^* \omega = 0$ and deduce that $n_+ = n_-$. 

**Problem 4** The relevance of this problem is not meant to be clear yet, but will come up eventually. An open cover $U_i$ of $\mathbb{R}$ is called “locally finite” if, for any $x \in \mathbb{R}$, there are only finitely many $U_i$ containing $x$.

(a) Give an example of an open cover of $\mathbb{R}$ with no locally finite sub cover.

(b) Show that a locally finite cover of $\mathbb{R}$ contains only countably (or finitely) many open sets. (Hint: Every nonempty open set contains a rational point.)