Problem Set 6 – due Tuesday March 24

See the course website for policy on collaboration.

Problem 1 Let $I$ denote the closed interval $[-1,1]$ in $\mathbb{R}$. Let $X$ and $Y$ be compact manifolds which can be embedded in $\mathbb{R}^N$ for some $N$. (I am only embedding them because we have not yet talked about differential forms and Stokes theorem for abstract manifolds.) Let $\alpha$ and $\beta$ be two smooth maps $X \to Y$. The maps $\alpha$ and $\beta$ are defined to be homotopic if there is a smooth map $\phi : X \times [-1,1] \to Y$ such that $\phi(x,1) = \alpha(x)$ and $\phi(x,-1) = \beta(x)$.

(a) Show that, if $\omega$ is a closed dim $X$ form on $Y$, and $\alpha$ and $\beta$ are homotopic, then $\int_X \alpha^* \omega = \int_X \beta^* \omega$.

(b) Let $X$ be the circle $S^1$ and let $Y$ be the torus $S^1 \times S^1$. Choose a point $p \in S^1$. Let $\alpha$ and $\beta$ be the maps $\alpha(\theta) = (\theta,p)$ and $\beta(\theta) = (p,\theta)$. Show that $\alpha$ and $\beta$ are not homotopic.

(c) Let $X$ be the circle $S^1$ and let $Y$ be $\mathbb{R}^2 \setminus \{(0,0)\}$ Let $\alpha(\theta) = (\cos \theta, \sin \theta)$ and let $\beta(\theta) = (2 + \cos(\theta), \sin(\theta))$. Show that $\alpha$ and $\beta$ are not homotopic.

(d) Let $X$ be the circle $S^1$ and let $Y$ be $\mathbb{R}^2 \setminus \{(1,0),(-1,0)\}$ Let $\alpha(\theta) = (1 + \cos \theta, \sin \theta)$ and let $\beta(\theta) = (-1 + \cos \theta, \sin \theta)$. Show that $\alpha$ and $\beta$ are not homotopic.

Problem 2 Let $X$ be the one dimensional manifold $\mathbb{R} \times \{-1,1\} \subset \mathbb{R}^2$. Define an equivalence relation $\sim$ on $X$ where $(x,1) \sim (x,-1)$ for $x \neq 0$, but $(0,1) \not\sim (0,-1)$. There are no other nontrivial equivalences (of course, every point is equivalent to itself.) Define $Y = X/\sim$. We’ll write $(x,\pm 1)$ for the equivalence class of $(x,\pm 1)$.

(a) Show that, if $A$ is any open subset of $Y$ containing $(0,1)$ and $B$ is any open subset of $Y$ containing $(0,-1)$, then $A \cap B \neq \emptyset$. The term for this is that $Y$ is not Hausdorff.

(b) Show that $Y$ is a one dimensional topological manifold. Recall that this means that, for any $y \in X$, there is an open set $W$ of $Y$ with $W \ni y$, an open subset $P \subset \mathbb{R}$ and a homeomorphism $\alpha : P \to W$.

Define $U \subset Y$ to be the set of points of the form $(x,1)$ for $x \in \mathbb{R}$ and let $V \subset Y$ be the set of points of the form $(x,-1)$.

(c) Show that there do not exist continuous functions $f$ and $g$ on $X$ such that $f + g = 1$ and such that $y \in Y : f(y) \neq 0 \subseteq U$ and $y \in Y : f(y) \neq 0 \subseteq U$. (The bar on top means to take the closure.) In other words, $Y$ does not have partitions of unity.

Let $P$ and $Q$ be the interval $(-1,1) \subset \mathbb{R}$. Define $\alpha : P \to Y$ to be the map $\alpha(x) = (x,1)$ and define $\beta : Q \to Y$ to be the map $\beta(x) = (x,-1)$.

(d) Show that we have smooth transitions between $(\alpha, P, U)$ and $(\beta, Q, V)$. In other words, $Y$ is a smooth manifold.

Problem 3 We write $\mathbb{C}$ for the complex numbers. The point of this problem is to construct a smooth manifold known as $\mathbb{CP}^2$. (There is a completely analogous $\mathbb{CP}^n$ for any $n$.)

Let $\mathbb{C}^4$ be the two dimensional complex vector space and let $S = \mathbb{C}^2 \setminus \{(0,0,0)\}$. Define an equivalence relation on $S$ by $(x_1,y_1,z_1) \sim (x_2,y_2,z_2)$ if there is a nonzero $c \in \mathbb{C}$ such that $(x_1,y_1,z_1) = c(x_2,y_2,z_2)$. As a topological space, $\mathbb{CP}^2 = S/\sim$.

Define the sets $U = \{(x,y,z) : x \neq 0\}$, $V = \{(x,y,z) : y \neq 0\}$ and $W = \{(x,y,z) : z \neq 0\}$ in $S$.

(a) Show that $U/\sim, V/\sim$ and $W/\sim$ are open in $\mathbb{CP}^2$ and $\mathbb{CP}^2 = (U/\sim)\cup (V/\sim)\cup (W/\sim)$.

Define maps $\alpha : \mathbb{C}^2 \to U/\sim$, $\beta : \mathbb{C}^2 \to V/\sim$ and $\gamma : \mathbb{C}^2 \to W/\sim$, sending $(s,t)$ to the equivalence classes of $(1,s,t)$, $(s,1,t), \text{and } (s,t,1)$.

(b) Show that $\alpha$ is a continuous bijection $\mathbb{C}^2 \to U/\sim$ with continuous inverse.

(c) Compute $\alpha^{-1}(U/\sim \cap (V/\sim))$ and $\beta^{-1}(U/\sim \cap (V/\sim))$.

(d) Write down an explicit formula for the map $\beta^{-1} \circ \alpha$ from the open set $\alpha^{-1}(U/\sim \cap (V/\sim))$ to the open set $\beta^{-1}(U/\sim \cap (V/\sim))$.

Congratulations, you have put the structure of a smooth manifold on $\mathbb{CP}^2$!