Determinants multiply

Let A and B be two $n \times n$ matrices. The point of this note is to prove that $\det(AB) = \det(A)\det(B)$. The textbook gives an algebraic proof in Theorem 6.2.6 and a geometric proof in Section 6.3. Our proof, like that in Theorem 6.2.6, relies on properties of row reduction.

I'll write $\Delta(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n)$ for the determinant of the $n \times n$ matrix with rows $\vec{w}_1$, $\vec{w}_2$, …, $\vec{w}_n$.

We begin by checking some special cases:

Special Case 1: The statement is true if $\det B = 0$. Proof: If $\det B = 0$ then B is not injective; say $B\vec{v} = \vec{0}$ with $\vec{v} \neq \vec{0}$. Then $AB\vec{v} = \vec{0}$ so AB is not injective and $\det(AB) = 0$.

Special Case 2: The statement is true if $\det A = 0$. Proof: If $\det A = 0$ then A is not surjective. Suppose that $\vec{w}$ is not in the image of A. Then $\vec{w}$ is also not in the image of AB, so AB is not surjective and $\det(AB) = 0$.

Special Case 3: A diagonal matrix. Let D be a matrix with d_1, d_2, …, d_n on the diagonal and zeroes off the diagonal. So $\det(D) = d_1 d_2 \cdots d_n$. I claim that $\det(DB) = d_1 d_2 \cdots d_n \det B$. Proof: Let the rows of B be $\vec{w}_1$, $\vec{w}_2$, …, $\vec{w}_n$. Then the rows of DB are $d_1\vec{w}_1$, $d_2\vec{w}_2$, …, $d_n\vec{w}_n$. So we want to show that

$$\Delta(d_1\vec{w}_1, d_2\vec{w}_2, \ldots, d_n\vec{w}_n) = d_1 d_2 \cdots d_n \Delta(\vec{w}_1, \ldots, \vec{w}_n).$$

This follows from the basic properties of determinant (specifically, Theorem 6.2.3.(a) in the book).

Special Case 4: A matrix with one off-diagonal entry. Let E be a matrix which looks like

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & k & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with the k entry in position (i, j). (So the example is $i = 4$, $j = 2$.)Then $\det E = 1$. I claim that $\det(EB) = 1 \cdot \det(B)$. Proof: Let the rows of B be $\vec{w}_1$, $\vec{w}_2$, …, $\vec{w}_j$, …, $\vec{w}_i$, … $\vec{w}_n$. Then the rows of EB are $\vec{w}_1$, $\vec{w}_2$, …, $\vec{w}_j$, …, $\vec{w}_i + k\vec{w}_j$, … $\vec{w}_n$. We have

$$\begin{aligned} \det(EB) &= \Delta(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_j, \ldots, \vec{w}_i + k\vec{w}_j, \ldots \vec{w}_n) \\ &= \Delta(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_j, \ldots, \vec{w}_i, \ldots \vec{w}_n) + k\Delta(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_j, \ldots, \vec{w}_j, \ldots \vec{w}_n) \\ &= \Delta(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_j, \ldots, \vec{w}_i, \ldots \vec{w}_n) + 0 \qquad = \det(B) \end{aligned}$$

Going from the first line to the second line we have used the linearity of determinant; going from the second line to the third line we have used that the row $\vec{w}_j$ occurs twice.

Special Case 5: A matrix which switches two rows. Let S be a matrix which looks like

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with the off diagonal entries in positions (i, j) and (j, i). (So the example is $i = 2$, $j = 5$.) Then $\det S = -1$. I claim that $\det(SB) = -\det(B)$. Proof: Let the rows of B be $\vec{w}_1$, $\vec{w}_2$, …, $\vec{w}_j$, …, $\vec{w}_i$, … $\vec{w}_n$. Then the rows of SB are $\vec{w}_1$, $\vec{w}_2$, …, $\vec{w}_i$, …, $\vec{w}_j$, … $\vec{w}_n$. Our claim is the standard fact that

$$\Delta(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_j, \ldots, \vec{w}_i, \ldots \vec{w}_n) = -\Delta(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i, \ldots, \vec{w}_j, \ldots \vec{w}_n).$$

Using row reduction to compute determinants

Let A be an invertible matrix. Using the row reduction procedure, I claim that A can be written as a product of matrices which look like the matrices D, E and S discussed in Special Cases 3, 4 and 5.

An example is clearer than a proof. Let $A = \left(\begin{smallmatrix} 2 & 3 \\ 5 & 6 \end{smallmatrix}\right)$. In the left columns below I'll run row reduction; in the right columns I'll start building an expression for A as a product of matrices of the various special forms.

$$\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3/2 \\ 5 & 6 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3/2 \\ 5 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3/2 \\ 5 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3/2 \\ 0 & -3/2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3/2 \\ 0 & -3/2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3/2 \\ 0 & -3/2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3/2 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3/2 \end{pmatrix} \begin{pmatrix} 1 & 3/2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3/2 \\ 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3/2 \end{pmatrix} \begin{pmatrix} 1 & 3/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In short, every time that we add one row to another, we pick up an E. Every time we rescale a row, we pick up a D. If we had switched two rows, we would have gotten an S. At the end of the day, we have $A = X_1 X_2 \dots X_m \mathrm{Id}$, where each X_i is either a D-matrix, an E-matrix or an S-matrix. Notice that we know the row reduction ends at the identity, since A is invertible.

Using special cases 3, 4 and 5, we know that know that

$$\det(A) = \det(X_1 X_2 \dots X_m \mathrm{Id}) = \det(X_1)\det(X_2 \cdots X_m \mathrm{Id}) = \det(X_1)\det(X_2)\det(X_3 \cdots X_m \mathrm{Id}) = \cdots$$
$$= \det(X_1)\det(X_2)\cdots\det(X_m)\det(\mathrm{Id}) = \det(X_1)\det(X_2)\cdots\det(X_m)\cdot 1 = \det(X_1)\det(X_2)\cdots\det(X_m).$$

For example, we are saying that

$$\det \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = \det \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \det \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \det \begin{pmatrix} 1 & 0 \\ 0 & 3/2 \end{pmatrix} \det \begin{pmatrix} 1 & 3/2 \\ 0 & 1 \end{pmatrix}$$

or, in other words, that

$$\det \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = 2 \times 1 \times (-3/2) \times 1.$$

Your book has a very similar discussion in Algorithm 6.2.

The proof

Our goal is to show that $\det(AB) = \det(A)\det(B)$. If A is not invertible, we already checked this in Special Case 2. Otherwise, write $A = X_1 X_2 \cdots X_m \mathrm{Id}$ with each X_i either a D-matrix, an E-matrix or an S-matrix. As we just discussed,

$$\det(A) = \det(X_1)\det(X_2)\cdots\det(X_m).$$

Using Special Cases 3, 4 and 5 over and over again,

$$\det(AB) = \det(X_1 X_2 \dots X_m B) = \det(X_1)\det(X_2 \cdots X_m B) =$$
$$\det(X_1)\det(X_2)\det(X_3 \cdots X_m B) = \cdots = \det(X_1)\det(X_2)\cdots\det(X_m)\det(B) = \det(A)\det(B).$$

This is what we wanted to prove.