Let $V$ be a complex vector space equipped with a positive definite Hermitian form $B(\ ,\ )$. We check some of the properties of $B(\ ,\ )$ which were asserted without proof on Tuesday:

**Problem 1.** Let $\vec{v} \in V$. You might wonder whether we should define $\vec{v}^{\perp}$ to be $\{\vec{w} : B(\vec{v},\vec{w}) = 0\}$ or $\{\vec{w} : B(\vec{w},\vec{v}) = 0\}$. Show that it doesn’t matter, because these are the same subspace.

**Problem 2.** Let $W$ be a subspace of $V$. Show that $W \cap W^{\perp} = \{\vec{0}\}$.

Recall that $A : V \to V$ is called:
- **Hermitian** if $A = A^\dagger$
- **unitary** if $A^{-1} = A^\dagger$.
- **normal** if $AA^\dagger = A^\dagger A$.

**Problem 3.** Let $V \cong \mathbb{C}^2$ with the standard Hermitian inner product. Let $A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$.

(1) Show that $A$ is normal.
(2) Compute an orthonormal eigenbasis for $A$.
(3) Is $A$ Hermitian? Is it unitary?

**Problem 4.** Let $V$ be a vector space with a positive definite Hermitian form. Let $T : V \to V$ be a linear operator.

(1) If $T$ is normal, show that there are Hermitian operators $X$ and $Y$ with $T = X + iY$ and $XY = YX$.
(2) Conversely, show that, if $X$ and $Y$ are Hermitian operators $X$ and $Y$ with $T = X + iY$ and $XY = YX$ then $T$ is self-adjoint.

We now come back to some old problems from homework:

**Problem 5.** On Problem Set 10, Problem 3, we considered the vector space of functions $[-\pi, \pi] \to \mathbb{R}$ with the inner product $\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$. Now that we know about Hermitian forms, consider instead complex valued functions $[-\pi, \pi] \to \mathbb{C}$ with the inner product $\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x)\bar{g}(x)dx$.

(1) Recall that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Show that the list of functions $\frac{1}{\sqrt{2\pi}} e^{in\theta}$, for $n \in \mathbb{Z}$, is orthonormal.
(2) Convert the change of basis matrix between this orthonormal basis, and the orthonormal basis $\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(n\theta), \frac{1}{\sqrt{\pi}} \sin(n\theta)$ on the Problem Set.

**Problem 6.** On Problem Set 8, Problem 3, we considered a differential equation for a collection of masses connected by springs. For simplicity, we take all the masses to be equal to $m$. Let $k_{ij}$ be the spring constant on the spring joining mass $i$ to mass $j$. Then the equations of motion are

$$m \frac{d^2x_i(t)}{dt^2} = \sum_j k_{ij}(x_j(t) - x_i(t)).$$

(1) Rewrite this equation in the form $m \frac{d^2}{dt^2} \vec{x}(t) = -K \vec{x}(t)$ for a matrix $K$. As a starting example, you might want to do the case where there is a spring from mass 1 to mass 2 of strength $k_{12}$ and a spring from mass 2 to mass 3 of strength $k_{23}$.
(2) Show that the eigenvalues of $K$ are nonnegative real integers.
(3) Show how to convert eigenvalues of $K$ into solutions of the form $\vec{x}(t) = \vec{a} \cos(\omega t)$ for some vector $\vec{a}$. 