More on transposes, orthogonal complement
**Theorem** The rank of a matrix equals the rank of its transpose. Recall that $\text{rank}(A) = \dim \text{Image}(A)$.

We proved this before using row and column reduction; let’s give a new proof without coordinates.
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Orthogonal complement
We can use the dual to build something like orthogonal complement without working over the field $\mathbb{R}$. Let $V$ be a vector space and let $W$ be a subspace of $V$. Then we set $W^\perp$ to be the subspace of $V^*$ defined by

$$W^\perp = \{ v^* \in V^* : v^* \text{ is } 0 \text{ on } W \}.$$

In other words, $W^\perp = \text{Ker}(V^* \to W^*)$. (Your book uses $W^\circ$.)
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**Wake up question:** If \(\dim V\) is finite, then we have \(\dim W^\perp = \dim V - \dim W\).

**Proof:** Since \(W \to V\) is injective, the map \(V^* \to W^*\) is surjective. By rank-nullity, \(\dim \ker(V^* \to W^*) = \dim V^* - \dim \text{Image}(V^* \to W^*) = \dim V^* - \dim W^* = \dim V - \dim W\). \(\square\)
Time for you to talk!

**Problem 1** Let $X \subseteq Y \subseteq V$. Show that $X^\perp \supseteq Y^\perp$ (these are both subspaces of $V^*$).

**Problem 2** Let $W \subset V$ be vector spaces. Show that $(W^\perp)^\perp \supseteq W$.

This one is a bit broken: $(W^\perp)^\perp$ is in $V^{**}$, not in $V$. So this only makes sense if we identify $V$ and $V^{**}$, which only works in finite dimensions, or if we ask that the natural map $V \to V^{**}$ carries $W$ into $(W^\perp)^\perp$, which is true.

**Problem 3** Let $W \subset V$ be finite dimensional vector spaces. Show that $(W^\perp)^\perp = W$.

**Problem 4** Let $X$ and $Y$ be subspaces of $V$. Show that $(X + Y)^\perp = X^\perp \cap Y^\perp$. If $V$ is finite dimensional, show also that $(X \cap Y)^\perp = X^\perp + Y^\perp$.

**Problem 5** Let $V$ and $W$ be vector spaces and let $A : V \to W$ be a linear transformation. Then $\text{Ker}(A^*) = \text{Im}(A)^\perp$. If $V$ and $W$ are finite dimensional, we also have $\text{Im}(A^*) = \text{Ker}(A)^\perp$. 