Problem 1. Consider the matrix

\[ A = \begin{bmatrix} 2 & 6 & 2 & 2 & 4 & 2 \\ 1 & 3 & 2 & 3 & 5 & -1 \\ -1 & -3 & -1 & -1 & -2 & 1 \\ 1 & 3 & 1 & 1 & 2 & -2 \end{bmatrix}. \]

The row reduction (reduced row-echelon form) of \( A \) is

\[ \begin{bmatrix} 1 & 3 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \]

(1) Compute a basis for the image of \( A \).
(2) Compute a basis for the kernel of \( A \).

Problem 2. Find a basis for the vector space of real polynomials of degree \( \leq 3 \) obeying \( f(1) = f(-1) = 0 \).

Problem 3. Let \( \vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \), \( \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( \vec{z} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). What are the coordinates of the vector \( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \) in the basis \( \vec{x}, \vec{y}, \vec{z} \)?

Basic proofs

Problem 4. Let \( V \) be a vector space. Prove, directly from the axioms of a vector space, that \( -(-\vec{v}) = \vec{v} \) for all \( \vec{v} \in V \).

Problem 5. Let \( V \) be a vector space over a field \( F \). Let \( a \in F \) and let \( \vec{v} \in V \) and suppose that \( a\vec{v} = \vec{0} \). Show that either \( a = 0 \) or \( \vec{v} = \vec{0} \) (or both). In addition to the axioms of a vector space, you may use that \( c\vec{0} = \vec{0} \) and \( 0\vec{x} = \vec{0} \).

Problem 6. Let \( V \) be a vector space and let \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) be linearly independent vectors in \( V \). Let \( \vec{w} \) be an additional vector. Show that \( (\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}) \) is linearly dependent if and only if \( \vec{w} \in \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) \).

Problem 7. Let \( F \) be a field and let \( F^\infty \) be the vector space of infinite sequences \( (a_1, a_2, a_3, \ldots) \) of elements of \( F \). Let \( \vec{e}_i \) be the element \( (0, 0, 0, \ldots, 1, 0, \ldots) \) of \( F^\infty \), whose single one is in the \( i \)-th position; let \( \vec{f} \) be the vector \( (1, 1, 1, 1, \ldots) \).

(1) Show that the infinite list of vectors \( \vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots, \vec{f} \) is linearly independent.
(2) Give (and prove your answer correct) a vector \( \vec{g} \) which is not in the span of \( \vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots, \vec{f} \).

Problem 8. Let \( V \) be a finite dimensional vector space and let \( X \) and \( Y \) be subspaces with \( X \cap Y = \{0\} \). Show that \( \dim V \geq \dim X + \dim Y \).
Challenging Proofs

Problem 9. Let $V$ be a vector space over a field $F$ and let $X$ and $Y$ be subspaces of $V$. Suppose that $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_a$ is a basis of $X \cap Y$, that $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_a, \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_b$ is a basis of $X$ and that $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_a, \vec{y}_1, \vec{y}_2, \ldots, \vec{y}_c$ is a basis of $Y$. Show that the list of vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_a, \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_b, \vec{y}_1, \vec{y}_2, \ldots, \vec{y}_c$ is linearly independent.

Problem 10. Let $V$ be a vector space and let $T : V \to V$ be a linear transformation obeying $T^2 = T$. Let $K = \text{Ker}(T)$ and let $I = \text{Image}(T)$. Show that $V = K \oplus I$.

Problem 11. Let $V$ be a finite dimensional vector space and let $T : V \to V$ be a linear transformation. Show that $\dim \text{Ker}(T^2) \leq 2 \dim \text{Ker}(T)$.